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No-Arbitrage Condition of Option Implied Volatility and Bandwidth Selection

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ABSTRACT A standard approach to option pricing is based on Black-Scholes type (BS hereafter) models utilizing the no-arbitrage argument of complete markets. However, there are several crucial assumptions, such as that the option underlying log-returns follow normal distribution, there is unique and deterministic riskless rate as well as the volatility of underlying log-returns. Since the assumptions are generally not fulfilled, the BS-type models mostly provide false results. A common market practice is therefore to invert option pricing model and using market prices of highly liquid options to get a so called implied volatility (IV). The BS model at one time moment can be related to the whole set of IVs as given by maturity/moneyness relation of tradable options. One can therefore get IV curve or surface (a so called smirk or smile). Since the moneyness and maturity of IV often do not match the data of valuated options, some sort of estimating and local smoothing is necessary. However, it can lead to arbitrage opportunity, if no-arbitrage conditions on state price density (SPD) are ignored. In this paper, using option data on DAX index, we analyze the behavior of IV and SPD with respect to different choices of bandwidth parameter *h* and a set of bandwidths, which violates no-arbitrage conditions, are identified. Moreover, it is documented that the change of *h* implies interesting changes in the violation interval of moneyness. We also show the impact of *h* on the total area of SPD under zero, which can be seen as a degree of no-arbitrage violation.

INTRODUCTION

One of the most challenging activities at the financial markets is the pricing and hedging of derivatives contracts. Here, the crucial step is selection of proper model to describe the behavior of returns of the underlying factors (mostly the underlying asset price). Long time ago, the practice was to assume Gaussian distribution as a reliable proxy to the empirical observations of stock price or FX rate returns and thus apply the famous Black and Scholes (1973) model to price an option.

Soon however, it was documented that the returns can be very far from the assumption of Gaussianity and thus the Black and Scholes model can be used only indirectly – take the market price of liquid option, invert the Black and Scholes formula, obtain a volatility (that is, implied volatility), put it into the formula by setting the parameters of illiquid option and get the price.

Obviously, the implied volatility is not a single number valid for any option, but rather discrete set of values obtained from all traded options that are sufficiently liquid. Since the illiquid options, we wish to price, or even exotic options, which we can trade only OTC, mostly have different parameters (moneyness, maturity) than those of traded options, some non-parametric smoothing (and extrapolation) is needed to estimate the implied volatility function.

Notwithstanding, the implied volatility function must be calculated carefully – there exist several conditions on the price of call and put options, that must be fulfilled. Otherwise an arbitrage opportunity can arise, that is, riskless profit higher than common riskless return.

Clearly, there exist many technics that can be used to adjust the observations and transform them into smooth function. For example, for this purpose Holcapek and Tichý (2011, 2012) recently applied a novel approach of fuzzy transform technique and obtain an implied volatility surface with interesting properties.

In this paper, in line with Benko et al. (2007), we apply relatively classic approach of local polynomial smoothing techniques and study the bandwidth selection process in more details. In particular, we change *h* and examine the impact on the interval of moneyness that brings arbitrage opportunity and on the total degree of no-arbitrage violation.

We proceed as follows. In the following section we briefly review the problem of option pricing. Next, we provide some basic facts about the implied volatility modeling. After that we describe the data used in this study and finally particular results are provided.

Option Valuation and the Concept of Implied Volatility

Options are non-linear types of financial derivatives, which gives the holder the right (but not the obligation) to buy the underlying asset in the future (at maturity time) at prespecified exercise price. Simultaneously, the writer of the option has to deliver the underlying asset if the holder asks.

Options can be classified due to a whole range of criteria, such as counterparty position (short and long), maturity time, complexity of the payoff function, etc. The basic features are *the underlying asset* (S), which should be specified as precisely as possible (it is important mainly for commodities), *the exercise price* (K), and *the maturity time* (T).

If the option can be exercised only at maturity time T, we call it the *European option*. By contrast, if it can be exercised also at any time prior the maturity day, that is, $t \in [0,T]$, we refer to it as *the American option*}. A special type of options, possible to be classified somewhere between European and American options is *the Bermudan option*, which can be exercised at final number of times during the option life.

In dependency on the complexity of the payoff function, we usually distinguish simple *plain vanilla options* (PV) and *exotic options*. However, by a plain vanilla option we generally mean call and put options with the most simple payoff function. Sometimes, by plain vanilla options we mean any option which is regularly traded at the market, that is, it is liquid and no special formula is needed to obtain its price.

Thus,
$\Psi_{call}^{vanilla} = (S_T - K)^+$
for vanilla call and
$\Psi_{put}^{vanilla} = (K - S_T)^+$
for vanilla put, where $(x)^+ \equiv \max(x, 0)$.

Due to the definition of an option – it gives a right, but not an obligation to make a particular trade – we can deduce basic differences between the short and the long position. While the payoff resulting from the long position is non-negative, either 0 or $S_T - K$, the payoff of the short position will never be positive, that is, it is either $K - S_T$ or 0. Moreover, it is obvious, that the long call payoff is not limited from above, but the short position payoff function goes only up to the exercise price (underlying asset price is zero).

Options are quite important type of financial derivatives since they allow to fit even very specific fears (hedging) and outlooks (speculation) about the future evolution. Due to the nonlinear payoff function and potential high sensitivity to changes in the input factors, such as volatility or even maturity, options are very challenging also for modeling purposes.

Obviously, since the standard option valuation model of Black and Scholes (1973) was based on the assumption of normally distributed returns, the presence of skewness and kurtosis at the market complicates the situation significantly. A common market practice is to use the market price as an exogenous variable to be put into the BS formula (Black and Scholes 1973). Thus, a so called implied volatility is obtained, that is, a number that assures that BS model provides the right price. Such implied volatility can subsequently be used to value even exotic options, which are not traded at the market.

Generally, the price of European option f at time t with maturity T and payoff function ψ is given by the payoff expected under risk neutral probabilities Q discounted by the risk less rate to the beginning (t), that is, by setting $t = T$ “ t:

$$f_t = e^{-r\tau} E_{t,T}^{Q} [\Psi_T]$$

since the payoff at maturity is obviously identical to the European option value at the same time.

For example, assuming the payoff function of plain vanilla call and the normal distribution we get the valuation formula as follows (*BS model for vanilla call*):

$$f_{call}^{vantilla}(\tau, S, K, r, \sigma, \ ) = S\, F_N(d_+) - e^{-r\tau} K\, F_N(d-)$$

Here, S is the underlying asset price at the valuation time (t) and it is supposed to follow log-normal distribution, t is the time to maturity, r is riskless rate valid over t, s is the volatility expected over the same period, both per annum, and $F_N(x)$ is distribution function for standard normal distribution.

If the price of some options is available from the market, we can invert the formula to obtain the implied volatility, that is, the number that makes the formula equal to market price. Besides the important works, whose authors analyzed the impact of implied volatility on option price, belongs, besides others Dupire (1994), who formulated a process followed by the underlying asset price in dependency on the moneyness and maturity, and Rubinstein (1994), who formulated a discrete time model, the implied binomial tree.

Obviously, the implied volatility will differ for various input data, especially due to the moneyness (relation of the spot price and exercise price) and the time to maturity – otherwise the model could not provide correct price. The dependency of the implied volatility on these two factors can be explained by the risk of jumps in the underlying asset price or other deviations from the assumption of Gaussianity. For example, Yan (2011) carefully analyzed the impact of jump risk on the slope of the implied volatility function, which is informally referred to as the smile, and showed some interesting relations between the returns and the slope.

Although there exist many various approaches for the construction of the volatility curve or surface, including some recent alternatives, such as the application of radial basis function (see for example, Glover and Ali (2011) and references therein), we follow here relatively conservative approach adopted by Benko et al. (2007).

DATA

In order to analyze the impact of the bandwidth *h* selection, let us assume in line with Benko et al. (2007) the market prices of call and put options on German stock index (DAX) collected on a given day – all options matures in 15 days – and extract the implied volatilities. The researchers choose the same data as Benko et al. (2007) in order to compare our results (for various bandwidth parameters) with their results.

It is apparent, that for far OTM/ITM options, there are important differences between the prices of call and put options. Clearly, such observation is not supported by the put call parity, though it can happen due to the non-Gaussianity of the log-returns of the underlying asset combined with non-symmetric risk attitude.

RESULTS

In this section the researchers follow the procedure suggested in Benko et al. (2007) in order to smooth the IV function and calculate the state price density. The researchers apply both proposed algorithms, the first one (unconstrained model) estimates the IV and SPD using classical local quadratic smoothing technique, while the second one (constrained model) enriches it by the no-arbitrage constraint on SPD. Specifically, the researchers change the bandwidth *h* when smoothing the IV curve and observe the impact on the state price density.

In particular, the researchers consider the implied volatilities of DAX options with the same maturity on a given day, see preceding section, as inputs in order to obtain the IV curve, that is, a function of volatility and moneyness. In order to smooth the data the researchers apply the non-parametric approach of kernel regression assuming Epanechnikov kernel function (Epanechnikov 1969):

$$K_h(x) = \frac{0.75}{h}\left(1 - \frac{x^2}{h^2}\right) I_{|x/h| \leq 1} \tag{5}$$

where *I* states the indicator function and *x* is the smoothed variable.

The researchers start with the same bandwidth as in the original analysis (see Benko et al. 2007) by selecting $h = 0.045$, see Figure 1. As one might assume, the highest density can be observed for moneyness around 1 – it approaches 10. It is natural, since the researchers have defined the state price density as the second derivative of the option pricing function with respect to the strike price; that is, the rate of option price change is clearly most sensitive to the strike price for ATM options. The unconstrained estimation (dash line) is negative in two intervals for the smallest values of moneyness. Increasing the moneyness, the no-arbitrage constraint is no more active and therefore the constrained estimation (solid line) coincides with the unconstraint one.

In this paper the researchers focus on whether the estimated state price densities correspond to no-arbitrage assumptions or not. In other words, the densities should always be non-negative. It is apparent that there are (at least) two intervals, for which the smoothing of implied volatility surface leads into negative values of state price densities. One of them is quite easy to be overcame, since it is given by balancing between very far OTM calls/puts; one might assume that the function might have similar behavior also for the right tail, that is, for very far ITM calls (there were no available data to examine this issue in more details). The second is

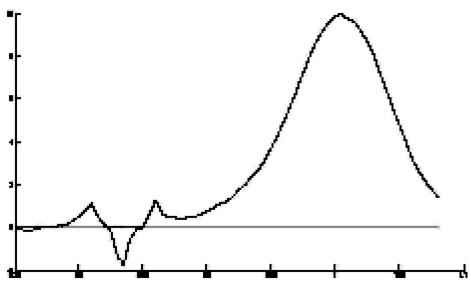


Fig. 1. State price density versus moneyness for DAX option prices assuming *h* = 0.045

probably given by a deficiency in the optimization algorithm since we can observe quite deep slump.

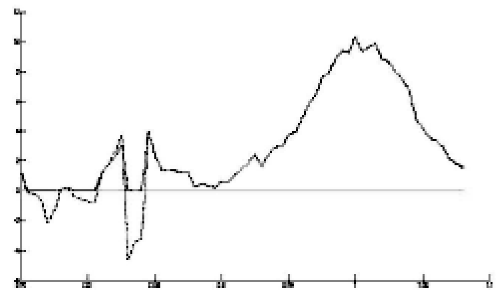
Next, we try to change the bandwidth *h* and observe what happens. In particular, we consider bandwidths between 0.030 and 0.050 with step length 0.001. Note, that due to the lack of data it does not make sense to evaluate the implied volatility function for *h* lower than 0.030 (undersmoothing); similarly, higher *h* than 0.05 would lead to over-smoothing.

Results for particular *h = 0.03, 0.035, 0.04, 0.045, 0.05* are apparent from Table 1 and Figure 2. One can easily observe that the lowest *h* leads to three intervals of moneyness allowing for arbitrage opportunity, each of them on the left tail, that is, OTM options. Although it is not easy to compare the intervals for particular *h*, the arbitrage area is strictly increasing when *h* decreases.

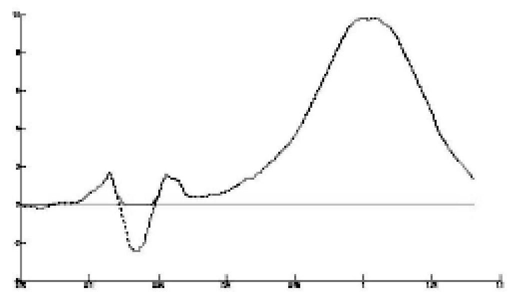
Table 1: Violations of no-arbitrage condition on SPD with different *h*

h	*Interval*	*Area (magnitude)*
0.030	0.755-0.775; 0.79-0.805; 0.83-0.84	0.090
0.035	0.75-0.775; 0.795-0.8; 0.825-0.845	0.071
0.040	0.75-0.77; 0.825-0.845	0.044
0.045	0.75-0.77; 0.825-0.85	0.022
0.050	0.75-0.775; 0.825-0.85	0.006

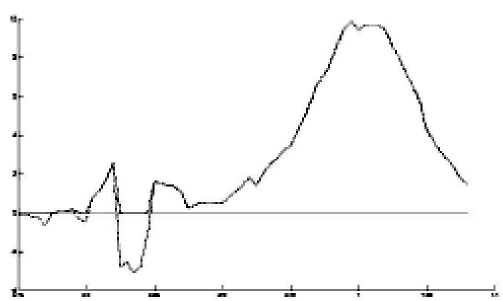
Finally, we summarize the results for all bandwidth choices at Figure 3 and Figure 4. The evolution of the moneyness intervals that violates no-arbitrage conditions is captured at Figure 3. We can observe three such intervals for h ε (0.03.0.037); two intervals for h ε (0.038,.052) and only one interval for $h = 0.053$) . Finally, for higher choices of bandwidth parameter, the SPD was non-negative everywhere. Figure 4 presents



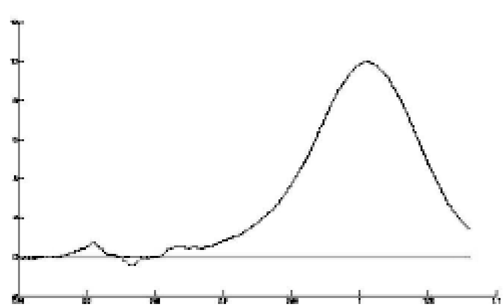
h=0.04



h=0.035



h=0.04



h=0.035

Fig. 2. State price density versus moneyness for DAX option prices assuming for various *h*

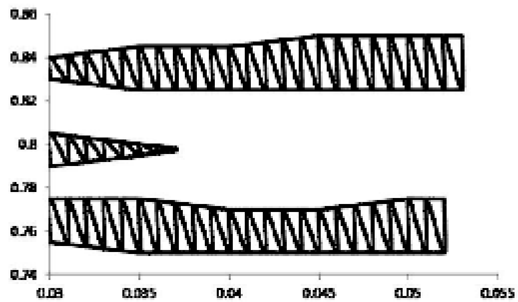


Fig. 3. The evolution of no-arbitrage violation intervals for h ε (003, 0.055)

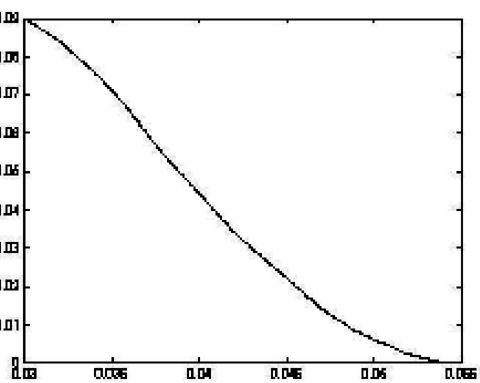


Fig. 4. The degree of no-arbitrage violation for h ε (003, 0.055)

the dependence of the magnitude of arbitrage areas on the bandwidth parameter.

CONCLUSION

In many cases, there is no way to valuate an option but to use implied volatility extracted from market prices. Since the moneyness and maturity of implied volatilities often do not match the data of valuated options, some sort of smoothing and interpolation is necessary. However, it can lead to arbitrage opportunity, if no-arbitrage conditions (non-negativity of SPD) are ignored. In this paper, we analyzed the behavior of SPD (state price density) with respect to changes in bandwidth parameter. Using option data on DAX index it was documented that the no-arbitrage violating intervals of moneyness as well as the total area of SPD under zero heavily depends on the choice of this parameter. The researchers observed that as the bandwidth parameter increases the degree of no-arbitrage violation decreases. Moreover, for $h > 0.053$ the no-arbitrage conditions were satisfied, because the intervals of violations disappeared.

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