PRINT: ISSN 0972-0073 ONLINE: ISSN 2456-6802
Anthropologist, 16(1-2): 1-5 (2013)
DOI: 10.31901/24566802.2013/16.1-2.01

An Analysis of In-service Teachers' Pedagogical Content Knowledge of Division of Fractions

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KEYWORDS Division. Fraction. In-service Teachers. Pedagogy. Mathematics Education

ABSTRACT The aim of the current study is to investigate in-service teachers' understanding of and difficulties for teaching division of fractions. Data were collected from 10 in-service middle school teachers (5 male, 5 female) from three different schools in Turkey, whose years of experiences range from 1 to 9 years. Six of the teachers were graduated from primary education departments whereas the rest were graduated from secondary school science and mathematics departments. Two-part division of fractions test was used to collect data. The first part of the test is to measure content knowledge while the second part is to measure pedagogical content knowledge. The results show that teachers had sufficient content knowledge to solve the problems in the first part of our test. However, they lacked the pedagogical content knowledge to teach division of fractions.

INTRODUCTION

The National Council of Teachers of Mathematics [NCTM] (2000) stated that middle-grade students should have a deep understanding of fractions. However, fractions are one of the most challenging concepts for students to master (Perie et al. 2005), and division is considered to be one of the most difficult operations (Ma 1999). Consequently, division of fractions is a difficult topic for students to understand (Bulgar 2003; Ma 1999). One of the main reasons for the challenges students encounter in developing conceptual understanding of division of fractions is that students are rarely provided with authentic problems for which they construct intuitive solutions (Bezuk and Armstrong 1993). Rather, the teaching of division of fractions is almost exclusively focuses on manipulation of arithmetic operations and memorizing a set of rules and procedures. As a result, the procedures they follow when dividing fractions often appear to be meaningless to students. Indeed, previous research in various countries showed that mathematics teachers lacked the necessary content knowledge to teach division of fractions. Although the teachers could successfully perform the operations using the traditional algorithm (invert-and-multiply), they could not provide an explanation why it worked (Ma 1999). In the current study, we examined Turkish mathematics teachers' pedagogical content knowledge (PCK) of division of fractions. Teachers answered a set of questions that addressed both the content knowledge and the PCK to teach fraction division.

Interpretations for Division of Fractions

Regardless of the methods used to solve division of fractions, the result would be meaningless without an interpretation (Ott et al. 1991). Two meaningful interpretations suggested by Ott et al. (1991) are (a) "the measurement meaning of division" (p. 11) and (b) "the partitive meaning of division" (p. 11). In the measurement meaning of division, the whole is known, and the problem entails finding the number of parts to cover the whole. In partitive interpretation, on the other hand, the number of parts is known, and the size of each part is to be found. Fundamentally, "how many ___ are in ___" is asked in measurement problems. For example, "how many ¼ are ¾ in "is a measurement problem, and it can also be given in a context: "How many quarter liter-sized bottles need to keep ¾ liter milk." The idea behind such questions is the same with natural numbers: the whole is divided by the size of the part. In this case, the solution to this question is , ¾ ÷ ¼,and it can be represented visually. Bulgar (2003) found that students could be able to find their own strategies using the solution methods they learned from natural numbers. Ott et al. (1991) investigated the understanding of partititve interpretation of division of fractions and provided several working examples. The idea of partititve meaning can be easily understood from a natural number example and then be extended to fraction scenarios. “Samantha has 6 cookies to be given to 2 friends. How many cookies does each friend get?” (Ott et al. 1991: 8). A contextual problem including rational numbers instead of natural numbers would be “if 2/3 of a glass can hold 2/5 liters of water, how much water one glass can hold?” The strategy follows finding how much water 1/3 of a glass (unit fraction) can hold water and then finding the amount of whole glass can contain. These two interpretations are to give meaning to the results of strategies used to solve problems. Besides giving meaningful contexts to students, mathematical justifications of division of fractions algorithm should be explained too.

Mathematical Justifications of Division-of-Fractions Algorithm

Inverse Operation: Division is expressed as the inverse operation of multiplication (Tirosh 2000), and the algorithm is showed by solving the equation as follows:

$$\frac{a}{b} \div \frac{c}{d} = A$$

$$\left(\frac{a}{b} \div \frac{c}{d}\right) x \frac{c}{d} = Ax \frac{c}{d}$$ *(Both sides are multiplied by the same fraction)*

$$\frac{c}{d} x \left(\frac{a}{b} \div \frac{c}{d}\right) = A\bar{x} \frac{c}{d}$$ *(Commutative property for fraction multiplication)*

$$\frac{a}{b} x \left(\frac{c}{d} \div \frac{c}{d}\right) = A\bar{x} \frac{c}{d}$$ *(Commutative property for fraction multiplication)*

$$\frac{a}{b} x (1) = Ax \frac{c}{d}$$ *(Identity property)*

$$\frac{a}{b} = Ax \frac{c}{d}$$ *(Identity property)*

$$\frac{a}{b} x \frac{d}{c} = Ax\left(\frac{c}{d}\right)x \frac{d}{c}$$ *(Both sides are multiplied by the same fraction)*

$$\frac{a}{b} x \frac{d}{c} = Ax\left(\frac{c}{d} x \frac{d}{c}\right)$$ *(Commutative property for fraction multiplication)*

$$\frac{a}{b} x \frac{d}{c} = Ax (1)$$ *(Inverse property for fraction multiplication)*

$$\frac{a}{b} x \frac{d}{c} = A$$ *(Identity property)*

$$A = \frac{a}{b} x \frac{d}{c}$$

$$\therefore A = \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} x \frac{d}{c}$$

Complex Fraction: With students’ knowledge of fraction multiplication and fraction equation, division of fraction algorithm can be explained (Li 2008).

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b}}{\frac{c}{d}} x\ 1 = \frac{\frac{a}{b}}{\frac{c}{d}} x \frac{\frac{d}{c}}{\frac{d}{c}} = \frac{\frac{a}{b} x \frac{d}{c}}{\frac{c}{d} x \frac{d}{c}} = \frac{\frac{a}{b} x \frac{d}{c}}{1} = \frac{a}{b} \square \frac{d}{c}$$

$$\therefore \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} x \frac{d}{c}$$

Common Denominator: Common denominator strategy follows similar method with addition and subtraction (Li 2008). In common denominator strategy, fractions first converted into like fractions.

$$\frac{a}{b} \div \frac{c}{d} = \frac{axd}{bxd} \div \frac{cxb}{dxb} = \frac{axd \div cxb}{bxd \div dxb} = \frac{axd \div cxb}{bxd \div bxd} = \frac{axd \div cxb}{1}$$

$$= \text{axd} \div \text{cxb} = \frac{axd}{bxc} = \frac{a}{b} x \frac{d}{c}$$

$$\therefore \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} x \frac{d}{c}$$

Instructional procedures presented here are invaluable aids for teachers to learn different approaches to teach division of fractions. However, most pre-service and in-service teachers still do not have adequate conceptual knowledge on division of fractions (Ma 1999).

Students’ Error Patterns on Division of Fractions

Tirosh (2000) proposed investigation of students’ error patterns on division of fractions in order to analyze teachers’ knowledge on the concept. Tirosh (2000) categorized children’s knowledge of division of fractions into three categories: (a) “algorithmically based mistakes” (p. 7), (b) “intuitively based mistakes” (p. 7), and (c) “mistakes based on formal knowledge” (p. 7).

In algorithmically based mistakes, students use the appropriate operations; however, they might have an error in one or more components of the strategy (Ashlock 2006). For example, in the division of fractions, a student could invert the divisor correctly, but instead of conducting multiplication of fractions, the student could cross multiply fractions. This error presents us that the student knows invert-and-multiply algorithm but has a misunderstanding of a step of it (Ketterlin-Geller and Yovanoff 2009).

Tirosh (2000) explains mistakes based on formal knowledge as “incorrect performance due to both limited conceptions of the notion of fraction and inadequate knowledge related to the properties of the operations” (p. 7). A student might solve a division of fractions problem with an incorrect formal knowledge of division being commutative. Thus, the student could solve 1/8÷ 1/4 as follows:

$$\frac{1}{8} \div \frac{1}{4} = \frac{1}{4} \div \frac{1}{8}$$ *(Commutative property)*

$$\frac{1}{4} \text{ x } \frac{8}{4} = \frac{8}{4} = 2$$

Ashlock (2006: 1) suggests becoming more "diagnostically oriented" in order to improve teaching mathematics. This suggestion guides teachers not only score student work but analyze student work in depth.

Objectives

The objective of the current study is to analyze in-service teachers' responses to the division of fractions test. It is aimed to investigate teachers' understanding of division of fractions and to figure out their difficulties in teaching of division of fractions to their students.

METHODS

Participants

The participants of the current study were 10 in-service middle school teachers (5 male, 5 female) from three different schools in Turkey. Four of the teachers were in their first year teaching experiences after they graduated from university. The most experienced teacher had 9 years of experience. Even though all teachers are teaching grades 5 through 8, four of them were graduated from secondary school science and mathematics programs, who were designed to educated high school teachers.

Instrument

To investigate in-service teachers' understanding of and difficulties for teaching division of fractions, a two-part test, which was developed by Li and Smith (2007), was conducted. The first part of the test was measuring teachers' content knowledge on division of fractions. The teachers were asked to solve six fraction division questions. The first three questions were given without context. The next two questions were word problems. One of the word problems was a measurement problem while the other was partititive (sharing). In the last question, participants are asked to compare $9/11 \div 2/3$ and $9/11 \div 3/4$ without any calculation and explain their reasoning.

The second part of the test was comprised of four questions to measure PCK of teachers. In the first question, a scenario was given and the teacher's response to it was asked. In the scenario, a student said multiplication of fraction was easy because you multiply denominators with denominators and numerators with numerators. Then, she suggested defining other operations with fractions with the same strategy as follows:

$$\frac{a}{b} + \frac{c}{d} = \frac{a+b}{b+d}$$

$$\frac{a}{b} - \frac{c}{d} = \frac{a-b}{b-d}$$

$$\frac{a}{b} \div \frac{c}{d} = \frac{a \div b}{b \div d}$$

The second question asked to teachers how they would explain two different division situations: (a) $2/3 \div 2 = 1/3$ and (b) $2/3 \div 1/6 = 4$. .

In the next question, teachers were asked how they would explain to their students if their students would ask the reason for converting a division problem into a multiplication and inverting the divisor fraction. That is, teachers should explain invert-and-multiply strategy to their students.

The last question consisted of three parts on the division problem: $3/4 \div 1/2 =?$ First, teachers were asked to list plausible error patterns students could do while they were solving the problem. Then, teachers needed to explain plausible reasons of those error patterns. In the last part of the last question, teachers were asked to write word problem that used "division of three fourths by one half."

RESULTS

All teachers were completely and correctly answered all questions. However, there were only two computational errors among all answers.

The first question presented a hypothetical situation in which a student suggested that she could use the strategy for multiplication of fractions to add, subtract, or divide fractions. The teachers were expected to provide an explanation for why the strategy for multiplying fractions would or would not work for addition, subtraction, or division. Even though, the teachers could give relatively comprehensive explanations for why the strategy would not work for addition and subtraction, they could not provide sufficient explanation for why it should work for division of fractions. The most common suggestion for addition and subtraction was using counter examples. For the case of division of fractions, two teachers wrote "this is the strategy we use in class," two others wrote "same logic with other operations," a few of them provided a general explanation of division, and three of them did not even give an explanation. In short, the teachers could not provide a substantive explanation for why we could divide across the numerators and across the denominators as a way to divide fractions.

The second question asked to teachers how they would explain two different division situations: (a) $2/3 \div 2 = 1/3$ and (b) $2/3 \div 1/6 = 4$. All but two teachers could answer the first part of this question. The most common strategy was using a visual representation of fraction blocks to represent division of a fraction by a whole number. For the second part of the question, seven out of the ten teachers could provide an explanation. All of the explanations for the second part involved visual representations. Two teachers generated contexts for two problems to explain how to get those results. Only one teacher first used the invert-and-multiply strategy to explain the equalities.

Another question required teachers to provide a substantive explanation of the invert-and-multiply method.

$$\left(\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}\right)$$

The explanations made by teachers were mostly either using simple examples or brief verbal explanations without details. The simple examples included either dividing two or half. Another teacher used division of rational numbers ($4 \div 2$) to explain the reason of invert-and-multiply method.

$$4 \div 2 \rightarrow 2$$

$$\frac{4}{1} \div \frac{2}{1} \rightarrow \frac{4 \div 1}{1 \div 2} \rightarrow \frac{4}{2} = 4 \div 2$$

The last question was about the division of 3/4 by 1/2. First part of this question was to list 6th grade students' plausible mistakes on the problem. Six out of 10 teachers expressed "finding the common denominator" as one of the students' possible mistakes even though finding common denominator is a sound strategy in solving fraction divisions. Among six teachers, only one mentioned that was not a wrong strategy but needed more time to solve. "Numerator is divided by numerator and denominator is divided by denominator" is mentioned as another plausible student mistake by four teachers.

$$\frac{3}{4} \div \frac{1}{2} = \frac{3 \div 1}{4 \div 2} = \frac{3}{2}$$

The last part of the question asked teachers to write word problem that used "division of three fourths by one half." Half of the teachers could not even write meaningful word problems.

DISCUSSION

In general, the results suggest that teachers have procedural knowledge on division of fractions. However, their conceptual understanding of the fraction division is not very strong. Even though, they sound explanations to the pedagogical questions, they still need to think and study more on division of fractions.

The results suggest teachers' understanding of fraction division is limited. In the case of addition and multiplication, they were sure to disagree with the student's suggestion. Because, it was easy to find a counter example in order to disprove her statement. However, the strategy in division of fraction was correct and needed a sound explanation. Teachers' brief explanations of the division strategy point out lack of their conceptual understanding of division of fraction.

Even though there were two teachers who could not answer how to explain $2/3 \div 1/6 = 4$, and three teachers who could not answer how to explain , rest of the teachers were provided substantive explanations for fraction divisions. The results suggest when a student asks for an explanation for a fraction division problem, teachers would try to give different explanations rather than just repeating the invert-and-multiply strategy.

However, the result for the next question emphasizes the urgent need for teacher educators focus on more conceptual understanding of division of fractions. Furthermore, when teachers were asked to provide plausible student errors for the next question, most of the answers that were provided by teachers were wrong indeed. For example, teachers said that finding common denominator in division of fractions was a plausible student error. However, finding common denominator is an arithmetically correct method to solve a fraction division problem. Similarly "Numerator is divided by numerator and denominator is divided by denominator" is mentioned as another plausible student mistake by four teachers, and this is another arithmetically correct strategy. These results suggest that teachers' conceptual knowledge on division of fractions is lacking.

CONCLUSION

Division of fractions is one of the most difficult concepts to understand not only for students but also for teachers. Even experienced teachers could have difficulties in providing substantive explanations for the reason of invert-and-multiply strategy. Teachers' avoidance of giving explanations for division of fractions strategy question exposes their incompetence in understanding the underlying reasons of division of fractions.

The results of current study alerts teacher educator to give great importance in discussion of division of fractions in their courses. Building division of fractions concepts on division of natural numbers could be an easy beginning and also could provide conceptual base for a smoother transition. Before moving to non-contextual, abstract problems, conceptual understanding should be built with different models including area models, length models, and set models. Using different strategies and models should be encouraged. Even, for some situations, real world examples may be required to explain division of fractions. Bringing real objects, like bread, apples, or milk, into the classroom and representing fractions with those would make more sense to some students.

RECOMMENDATIONS

The study provides important recommendations not only for in-service teachers but also for teacher educators and in-service teachers. Teacher educator should take the results of this study as an emergent need for more focus in division of fractions. Teachers' difficulties presented in this study could be used as a diagnostic tool to teach pre-service teachers in conceptual understanding of division of fractions. Different strategies for showing why invert-and-multiply method works and examples support conceptual understanding division of fractions presented in this paper are useful for teacher educators and pre- and in-service teachers.

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