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Students' Understanding of Solving a System of Linear Equations Using Matrix Methods: A Case Study

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ABSTRACT This paper focused on the written work of two students to questions based on the solution of a system of linear equations using matrix methods. The objective was to gauge the possible level of mathematical understanding of the students by using a framework that was arrived at. That framework was used to analyse the level of mathematical understanding of those students' written responses to the questions, with the focus on their use of symbolic language. It was found that the framework enabled the researcher to get a deeper insight into those students use of the symbolic language, used both in an instrumental role and also as a communicative function.

INTRODUCTION

This paper reports on a case study conducted on students' mathematical understanding of the solution of systems of linear equations using matrix methods. The study was conducted with two first year university students in the province of KwaZulu-Natal (South Africa) who were studying a module for which introduction to linear algebra was one of the components. One of the reasons for the study was the researcher's concern on how the students who studied the module were writing solutions to problems that they were expected to solve. Another reason was to determine whether the mental constructions referred to in APOS (action-process-object-schema) Theory could be used to assess the level of mathematical understanding of a student, in the context of written responses. The conceptual framework informed by the literature review led to the formulation of a framework that could guide the assessing of such mathematical understanding. This framework guided writing of the section, Findings and discussion. In the conclusion the main findings are summarized and two recommendations are proposed for mathematics lecturers.

Research Question

The research question for this study was: *What does an examining of the written responses of students to solving systems of linear equations using matrix methods reveal about their level of mathematical understanding?* The following sub-questions were formulated to help answer the research question: *(1) What is meant by mathematical understanding? (2) How could a students' written response to a problem be assessed for mathematical understanding?*

Literature Review

This section focuses on the following: Mathematical understanding; Insights from studies on linear algebra; Methods for solving a system of linear equations using matrix techniques.

Mathematical Understanding

To unpack what is meant by mathematical understanding it is important to know what is meant by mathematics. Both of these were focused on in detail by Maharaj (2015). The main ideas are summarized below. The views of Dubinsky (2010) and Godino (1996) as to what is mathematical knowledge and mathematics, were found to be useful. Dubinsky's assumption on mathematical knowledge is:

An individual's mathematical knowledge is his/her tendency to respond to perceived mathematical problem situations and their solutions by reflecting on them in a social context, and constructing or reconstructing mental structures to use in dealing with the situations.

This implies that crucial to an individual's acquiring of mathematical knowledge is the constructing and even reconstructing of mental structures for use when faced with problem situations. Godino's view is based on the following assumptions: a) Mathematics is a human activity involving the *solution of problematic situations*. Mathematics progressively emerges and evolves during the finding of responses or solutions to those external and internal problems situations. b) *Specific institutions or collectives* involved in studying such mathematical problems share both the problems and their solutions. c) The problem-situations and the solutions found are expressed in symbolic form, so mathematics is also a *symbolic language*. That symbolic language codes information using mathematical symbols. Further this language serves both an *instrumental role* (to keep track of one's thinking) and also a *communicative function* (to communicate thinking externally to others). The former is used when an individual deals with the human activity of finding a solution of problematic situations, while the later comes into play when the problems and their solutions are shared. d) Mathematics is a *logically organized conceptual system*. The constructing and reconstructing of mental structures for use when faced with problem situations (Dubinsky 2010) includes the use of symbolic language in mathematics to keep track of one's thinking and to communicate such thinking externally to others. The above assumptions imply that certain institutional or global conventions need to be followed, by both those who teach and study mathematics.

A number of educationists have written on understanding in general (for example Skemp 1976; Nickerson 1985; Sierpinska 1994), understanding in mathematics (for example Hiebert and Carpenter 1992) and how to access understanding in mathematics (for example Barmby et al. 2007). The following two types of understanding were identified by Skemp (1976): (1) *relational understanding* refers to knowing what to do and why, and (b) *instrumental understanding* refers to the use of rules without understanding. With regard to understanding in mathematics Hiebert and Carpenter (1992) and Barmby et al. (2007) viewed this as the building of structures or context. The number and strengths of the connections within and between these structures or context facilitate the transfer of prior knowledge to novel situations, which is essential since previously learned strategies are used to solve many new problems. Without such transfers each new problem will require a separate strategy (Stylianides and Stylianides 2007). The implication here is that without the suitably connected mental structures to facilitate the transfer of knowledge it would be impossible for one to become mathematically competent. The above imply that the following could serve as a useful starting point to examine mathematical understanding: (1) students' errors, (2) *connections* made between *symbols* and *symbolic procedures* and *corresponding referents*, (3) *connections* between *symbolic procedures* and *problem solving situations*, and (4) *connections* made between *different symbol systems*.

Insights from Studies on Linear Algebra

Various research studies (for example, Dorier 1990; Vinner 1991; Dorier and Sierpinska 2001; Zaslavsky and Shir 2005; Stewart and Thomas 2009; Britton and Henderson 2009; Maharaj 2015; Ndlovu and Brijlall 2015) focused on students' difficulties with regard to concepts central to the study of undergraduate mathematics. The solution of a system of linear equations using matrix methods is focused on in the study of linear algebra. Despite some efforts to improve the curriculum, the learning of linear algebra is challenging for many students and there is agreement that the teaching and learning of linear algebra is frustrating for both lecturers and students (Dorier and Sierpinska 2001). In their study of the relationship between theoretical thinking and high achievement in linear algebra Sierpinska et al. (2002: 1) noted that "linear algebra, with its axiomatic definitions … is a highly theoretical knowledge, and its learning cannot be reduced to practicing and mastering a set of computational procedures". Stewart and Thomas (2009) noted that although students may cope with the procedural aspects of solving a linear system of equations and manipulating matrices they struggle to understand the crucial conceptual ideas that underpin them. One of the reasons for this could be that although concepts are presented through a definition which is in a natural language, quite often the definition embodies a symbolic representation. So this symbolic representation codes certain ideas and this could be a stumbling block to students who are unable to decode the relevant ideas. Britton and Henderson (2009) refer to this as the *obstacle of formalism* theme. A further stumbling block is that those definitions form the foundation as a starting point for concept formation and deductive reasoning (Vinner 1991; Zaslavsky and Shir 2005) used to understand the statements and proofs of theorems. So while students are expected to overcome the formalism obstacle they are also required to learn and understand new definitions and theorems; these contribute to their difficulties. The study by Britton and Henderson (2009) found that recognizing the need to set out a general argument is only the preliminary step. They found in the next step students frequently embarked on meaningless manipulation of symbols and even including extracts from the solution of similar problems that they memorized from lectures. To compensate for such student difficulties at times questions set in examinations focus mainly on using techniques and following certain procedures rather than on understanding the concepts (Dorier 1990). This gives a false sense that many students attain 'success' in examinations based on such characteristics. Lecturers should guard against this. The implication here is that questions that are set should have a balance in their focus on procedures or techniques and understanding of concepts. The study by Ndlovu and Brijlall (2015) used APOS Theory as a framework to analyse 85 pre-service teachers' mental constructions of concepts in matrix algebra. Those researchers reported that: schemata for basic algebra and real numbers are required for the conceptual development of matrix algebra; proficiency in the use of terminology and symbolic language promotes the learning of matrix algebra. In this study the focus is on the students' written responses to questions on the solution of systems linear equations, which include the use of symbolic language.

Methods for Solving a System of Linear Equations Using Matrices

To understand and determine the methods for solving a system of linear equations using matrices that undergraduate students should be exposed to in an introduction to linear algebra, a study of some of the prescribed books and research articles was done (for example Tan 2005, 2014; Lay 2006; Lazebnik 1996; Babajee 2014). That study revealed three methods to solve a system of linear equations using matrix methods. These are:

The technique of row reduction of the augmented matrix representation to row echelon form. This technique could be used to solve any system of equations. The number of equations need not equal to the number of variables. To the augmented matrix row operations are applied so that the initial system is transformed to an equivalent system whose augmented matrix is in row echelon form. Since the systems are equivalent their solution is the same. So the row echelon form arrived at could be used to represent the equivalent system of equations and from that the deductions could be made. The system could have a unique solution or infinitely solutions or no solution.

Using the inverse of the coefficient matrix to solve a system of linear equations whose matrix representation is in the form AX=C. This technique could only be used when the number of linear equations in the system is equal to the number of variables and the inverse of the coefficient matrix exists. In the matrix representation of the system AX=C, A is the square matrix obtained from the coefficients of the variables, X is the column matrix which denotes the variables and C is the column matrix which denotes the constants to the right of the equal to sign when the equations are written in standard form. The idea is to multiply from the left throughout the equation by the inverse of matrix A, A^{-1}. So $AX=C \Leftrightarrow A^{-1}AX=A^{-1}C \Leftrightarrow X=A^{-1}C$. Then matrix multiplication is used to arrive at the unique solution to the system.

Cramers' Rule. Let AX=C be the matrix representation of a system of n linear equations in n variables, with the matrix of coefficients nonsingular. Then the system has a unique solution given by

$$x_1 = \frac{\det(A_1)}{\det(A)},\ x_2 = \frac{det(A_2)}{det(A)},\ x_3 = \frac{det(A_3)}{det(A)},\ \dots,\ x_n = \frac{det(A_n)}{det(A)}$$

where A_1 is the matrix obtained from A by replacing the entries in ith column of A by the respective entries of the column matrix C.

It should be noted that the terms used (for example augmented matrix, row operations, row echelon form) and the use of symbolism to describe the above methods contribute to what Britton and Henderson (2009) refer to as the *obstacle of formalism* theme that compound students' difficulties.

Conceptual Framework

This section focuses on: APOS mental structures; Framework to gauge mathematical understanding.

APOS Mental Structures

According to Dubinsky (2010) an individual requires suitable APOS (action-process-object-schema) mental structures to deal with problem situations. The brief descriptions of action, process, object and schema below are based on those given by Weller et al. (2009) and Maharaj (2010, 2013, 2014).

- o *Action:* A transformation is first conceived as an *action*, when it is a reaction to stimuli which an individual perceives as external. It requires specific instructions, and *the need to perform each step of the transformation explicitly*.
- o *Process:* As an individual repeats and reflects on an action, it may be *interiorized* into a mental *process*. A process is a mental structure that performs the same operation as the action, but *wholly in the mind of the individual*. Specifically, the individual can imagine performing the transformation without having to execute each step explicitly.
- o *Object:* If one becomes aware of a process as a totality, realizes that transformations can act on that totality and can actually construct such transformations (explicitly or in one's imagination), then we say the individual has *encapsulated* the process into a cognitive *object*.
- o *Schema:* A mathematical topic often involves many actions, processes and objects that need to be *organized and linked into a coherent framework*, called a *schema*. It is coherent in that it provides an individual with a way of deciding, when presented with a particular mathematical situation, whether the schema applies.

Framework to Gauge Mathematical Understanding

Normally in a written response a student is expected to represent his or her thinking externally so that others could follow the reasoning. This implies that there should be some framework to access a student's level of mathematical understanding when a response is given in a written format. A detailed description of how such a framework was arrived at appears in Maharaj (2015). Based on the literature review on mathematical understanding and the APOS mental structures the framework illustrated in Table 1 was arrived at and revised from Maharaj (2015); the italics indicate the revisions.

METHODOLOGY

This was informed by the literature review and conceptual framework. Using his knowledge of APOS Theory and the solution of systems of linear equations using matrix methods the researcher arrived at schemata to solve n linear equations in n unknowns, (see Figs. 1 and 2). These together with the framework arrived at in Table 1 were used to analyze the written responses of two students to two test questions on the solution of linear equations using matrix methods. The marked attempts of the students to the two questions were returned to them during a three hour tutorial session. The two students were randomly chosen and their consent was

Table 1: A framework to access mathematical understanding

Literature source and depth	*Mathematical understanding*	*APOS mental structures*
Skemp	Instrumental	At most at an action level
Nickerson,	Relational ·	
Hiebert and Carpenter,	• Connected to an existing network	• Process, object, schema
Dubinsky, Maharaj	• Number and strength of connections	• Connections between existing schema
Godino	Mathematical symbolism	
Hiebert and Carpenter	• Instrumental role	• Interactions among mental structures of actions, processes, objects, schema
Barmby, Harries, Higgins and Suggate	• Communicative function	• Execution of mental structures; *external evidence*
Dubinsky, Maharaj	• Logically organized conceptual system	Degree of development and interrelationships of schema; *within schema and between schemata*

obtained. Those two students were in a tutorial group of twenty students. Both agreed that their work could be analyzed, they also volunteered to be interviewed during the tutorial and even outside the tutorial times - provided the latter was convenient to them. After the written work was analyzed by the researcher interview questions were noted to clarify aspects or issues noted during the analysis of the written work in terms of the framework in Table 1. About an hour was used during the tutorial to interview both of the students. Both the students requested for the interviews to continue during their free times. Within three days of the first interviews additional half an hour sessions were scheduled with each of those students in the researcher's office. During all the interview sessions further interview questions were formulated based on the initial responses of the interviewees.

The initial analysis of each student's written work and their interview responses were then used to document the findings and discussions on them, which are given in the next section. There the written extracts and interview responses of those two students are referred to by S1 and S2; R denotes the researcher. An idea is generally better understood if an example is used to illustrate that idea. Since only two participants' sources were to generate the evidence, this could be viewed as a case study. A case study could be chosen to illustrate a general principle (Cohen et al. 2007).

FINDINGS AND DISCUSSION

The relevant findings and discussions are indicated under the following sub-headings: Question 1 and Question 2. Under each of those sub-headings the question that the student responded to is stated and relevant extracts of student(s) responses are given. Those extracts are discussed together with relevant student and researcher responses during the interviews.

Question 1 [Solving a System of Linear Equations Using Matrix Concepts]

Use matrices to determine the values of a for which the system of equations

$$\begin{aligned} x + 4y - 2z &= a \\ 3x + 5y \quad &= 5 \\ 2x + 15y - 10z &= 5 \end{aligned}$$

has i) infinitely many solutions, ii) no solution, and iii) a unique solution.

An examination of the written response of student S2 revealed she was able to correctly write down the augmented matrix for the system of equations and correctly apply row operations. This and Extract 1, which gives part of the student's written response to Question 1, indicate that the student was able to retrieve and execute some sort of a schema for the solving of a system of linear equations. That execution was based on the use of writing down the augmented matrix and using the row reduction technique

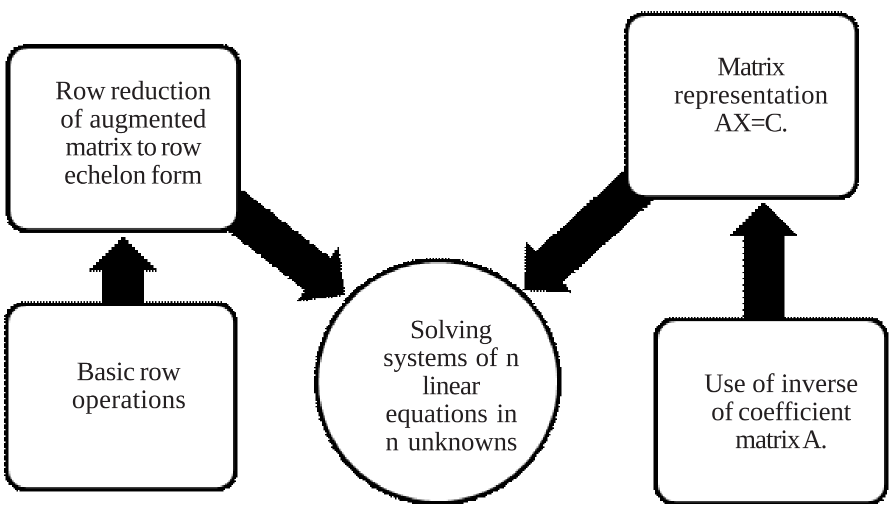


Fig.1. A schema to solve a system of n linear equations in n unknowns
Source: Author

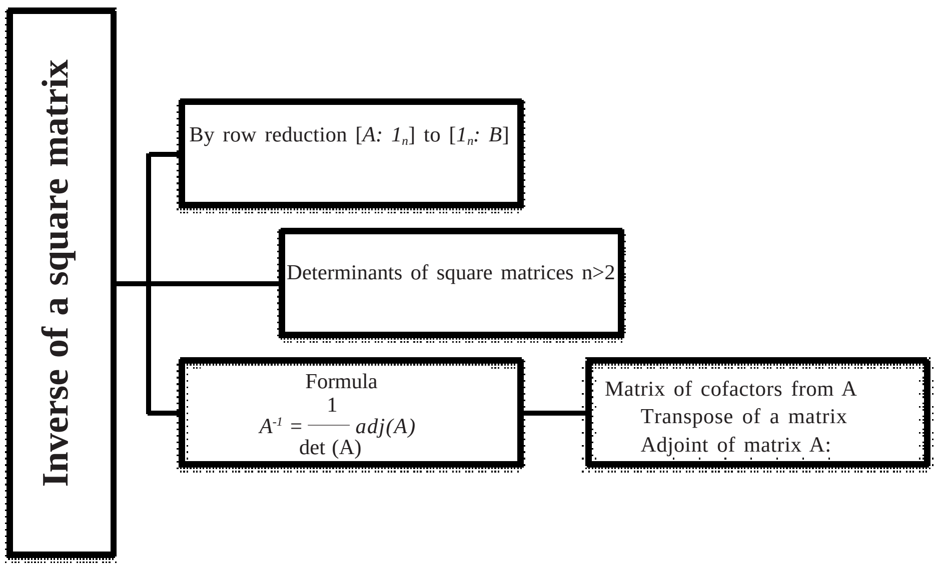
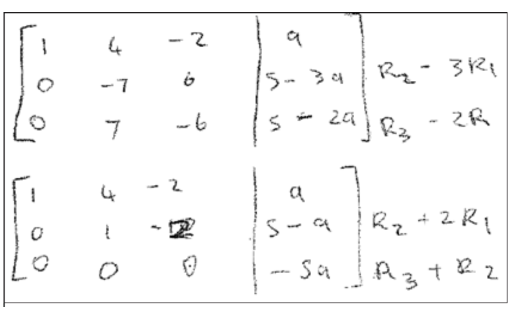


Fig. 2. Schema for inverse of a matrix
Source: Author

(see Fig. 1). The mathematical understanding displayed in the first augmented matrix in Extract 1 could be considered to be relational and the use of mathematical symbolism is both at the instrumental and communicative levels (see Table 1). The former indicates that there were connections among existing schema for the solution of a system of linear equations, using matrices.

Extract 1: Part of written response of student S2



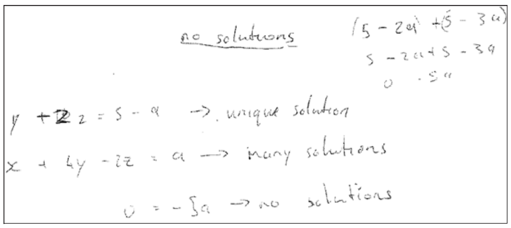
Note however the connections were insufficient, since the second augmented matrix in Extract 1 does not logically follow from the preceding one. There are two serious errors associated with the row operations communicated in the second and third rows of the second augmented matrix. In the second row, the row operation R_2+2R_1 is firstly illogical since it would take one further away from the concept of row echelon form of a matrix. Secondly the row operation communicated is not correctly executed, since that row operation would result in the following entries in the second row *[2 1 2: 5-a]* . Also note that in the third row of the second augmented matrix the student correctly communicates the row operation, R_3+R_2 . However, the execution is incorrect. These indicate that in arriving at the second augmented matrix the student's mathematical understanding of the selection of row operations and their execution with the intention of leading to the row echelon form of a matrix was instrumental; see Table1. So the mental structures were not even at most at an action level. The following transpired during the interview:

R: How did you get *-5a* here? *[pointing to entry to the right of vertical line in the 3rd row]*

S2: I subtracted instead of added in my mind …. answer should be *10-5a.*

Note that the student's explanation is not entirely correct. If she subtracted she would have arrived at the following entries in the last row *[0-14 12: -a]*. In Extract 2 the working on the right hand side *(5-2a) + (5-3a)*, indicates that she intended to add but it seems from the next step *(5-2a + 5-3a)* she subtracted the second 5 from the first 5. This suggests that she was unable to correctly interpret what she wrote down, which in turn implies her thinking was not consistent with the written step (5-2a + 5-3a).

Extract 2: Continuation of written response of student S2 to Question 1



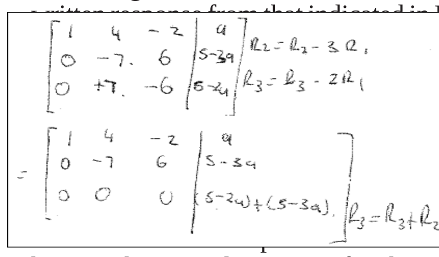
The written responses of students S1 and S2 for Question 1, see Extracts 1 and 3, revealed that both of them merely wrote down the augmented matrix related to the given system of equations and proceeded to use row operations. There was no evidence of an attempt to explicitly communicate the method they chose and their strategy to move towards a solution to the question. For example explicitly communicating these would be a write-up something to the effect: *We write down the augmented matrix and apply row operations to convert to row echelon form.* So for this aspect (explaining their thinking) both the students' written responses lacked a communicative function. This may seem trivial to some readers. However, it was this aspect of writing down one's thinking which led to the inability of student S2 to make meaningful conclusions from the augmented matrix she arrived at, see Extracts 2 and 1 respectively. The written response that followed Extract 1, see Extract 2, indicates the student could not use the second augmented matrix arrived at in Extract 1 to make any meaningful conclusions regarding the value(s) of that result in the required types of solutions. A meaningful conclusion from that augmented matrix could read as follows:

From this augmented matrix we get the following system of equivalent equations

$$x + 4y - 2z = a$$
$$y - 2z = 5 - a$$
$$0x + 0y + 0z = -5a$$

Such an explanation could help one to keep track of one's thinking and also the way forward with regard to valid conclusions. For example, from the third equation it could be concluded that if $a \neq 0$ then the system is inconsistent and will have no solution. These together with Extracts 4 and 5 suggest that for solving a system of linear equations, using row reduction applied to an augmented matrix, student S2 did not possess a logically organized conceptual system (see Table 1).

Extract 3 indicates that to solve the system of equations in Question 1, student S1 chose and executed the technique based on row reduction applied to the related augmented matrix. This extract together with Extract 4 also suggests that the student's mathematical understanding with regard to solving a system of linear equations; using the augmented matrix technique with suitable row operations; was at the relational level (see Table 1). The understanding of this technique to solve a system of linear equations seems to be connected to an existing network. In this network the number and strengths of connections seem to be sufficient. With regard to the APOS mental structures the Extracts 3 and 4 suggest that for this technique, the mental structures of this student included structures at the process, object and schema levels. Extract 4, which gives the continuation of the student's written response from that indicated in Extract 3,



[illegible] the pro- [illegible] t explic- [illegible] nt equa- [illegible] x in Ex- [illegible] dent was [illegible] mind to [illegible] tract 4 - [illegible] r this to happen the mental structure for the interpretation of the second augmented matrix in Extract 3 would have to be at the object level. For the written responses indicated in Extracts 3 and 4 the mental structures of actions, processes and objects would have to be coherently linked to a schema, and there would have to be connections between those existing schemata.

Extract 3: Part of written response of student S1

The use of mathematical symbolism by the student in Extracts 3 and 4 is clearly at an instrumental level. There is also evidence of that use serving a correct communicative function, for example in communicating the row operation of $R_2=R_2-3R_1$, in the second row of the first augmented matrix in Extract 3. The student is communicating that row 2 here is arrived at by taking the previous row 2 and subtracting three times the previous row 1. Note again the incorrect use of the equal to sign by this student in the context of equating the first augmented matrix with the second one. However, in Extracts 3 and 4 it is clear that her incorrect use of mathematical symbolism (the equal to sign) did not hinder her thinking. Rather to the reader of the solution it is more of an irritation. The conclusion that one could make is that this student's use of mathematical symbolism was not fully developed to serve a communicative function; see Table 1. These suggest that although student S1 could execute mental structures - the execution was not developed to a level to communicate externally to others without hindrances. The following transpired during the interview:

R: Is this correct*? [pointing to the '=' on the left hand side of the second matrix in Extract 3]*

S1: No … not supposed to have equal to sign

R: Why?

S1: It implies that this matrix *[pointing to the 1st one]* is equal to that matrix. *[pointing to the 2nd augmented matrix]*

R: What connective should be used?

S1: That … I don't know what it is called, wavey *[writes]*

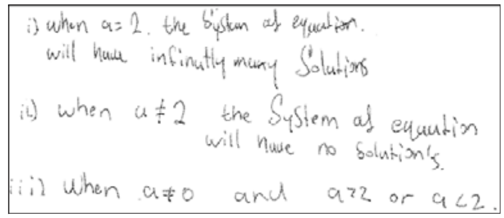
These suggest that the use of students' written work could be a useful source to clarify and even modify connections. If a student's attention is drawn to incorrect use of mathematical symbolism then this could help the student to clarify existing connections between schema and possibly be more careful in the use of mathematical symbolism at the communicative level. In trying to help the student to communicate better the following was focused on:

R: Is there anything missing from your work after this? *[pointing to the 2nd augmented matrix]*

After some prompting ….

S1: From the 3rd row we have 0x+0y+0z=10-5a.

Extract 4: Continuation of written response of student S1



An examination of the student's response to parts ii) and iii) of Question 1, see Extract 4, indicates contradictions. To get student S1 to focus on the contradictions the following transpired:

R: Look at this part of your answer. *[points to the part indicated as iii)]*

Student S1 was asked to illustrate the values of indicated for parts ii) and iii) on a number line. After some time she saw that the part was included in the answer for part ii); .

Question 2 [Use of Inverse Matrix to Solve a System of Linear Equations]

Let $A=\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -2 & 3 \end{bmatrix}$

i) Calculate det (A) , the determinant of A , by expanding along row 1. What can be concluded from this about the existence of the inverse of A?

ii) Evaluate A^{-1}, the inverse of A.

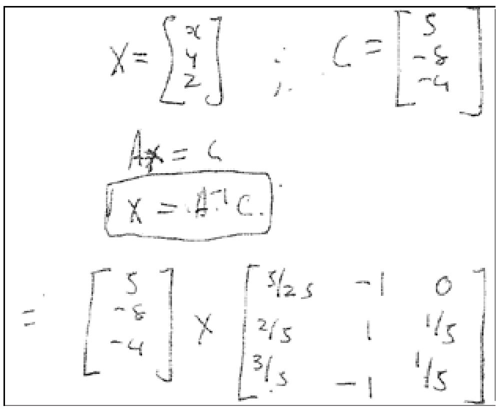
iii) Use A^{-1} to solve the following system of equations:

$$2x+y-z=5$$
$$-x-2y+3z=-8$$
$$-x-y+z=-4$$

A study of both students written responses indicated that they were able to calculate the determinant of the matrix A and make the conclusion that the inverse of A exists. Their written responses also indicated that both had suitable mental structures at least at an action level for the procedure to find the inverse of the matrix A, using row reduction. So both the students had some sort of a schema to find the inverse of a square matrix (see Figure 2). It was evident that both students understanding of the use of the inverse of a coefficient matrix to solve a system of n linear equations with n unknowns was at an instrumental level. Student S2 wrote down $AX=C$

followed by $A^{-1}AX=CA^{-1}$ which indicates the student was unable to link the property that matrix multiplication is not commutative to the matrix representation of a system of n equations in n unknowns, here n = 3. Further the use of mathematical symbolism was not developed to even an adequate instrumental role. The response of student S1 is given in Extract 5.

Extract 5: Written response of student S1 to third part of Question 2



Observe here that the student correctly wrote down the matrix X. However, the matrix representation does not does correspond to the given linear system of equations in part iii) of Question 2. To use from part ii) the system in part iii) has to be rewritten as an equivalent system whose coefficient matrix corresponds to the matrix A. Using this equivalent system the matrix representation could be arrived at. The following transpired during the interview:

R: Is the coefficient matrix of the given system *[pointing to part iii) of the question]* the same as the matrix A?

S1: No it is not

R: Can it be related to matrix A?

S1: I think it can.

When asked to explain the student was unable to continue. After a period of silence he did indicate the following: … *I saw an example similar to this … I didn't pay much attention to it.* It could be concluded from this that this student's schema to solve a system of n linear equations in n unknowns was not sufficiently developed. In particular the degree of interrelationships between schemata was lacking. The schema for using the inverse of a matrix (see Fig. 2) was not sufficiently connected to the relevant component of related matrix representation of an equivalent system in the schema for solving systems of linear equations using matrix methods (see Fig. 1). Notice also that the last line of the working in Extract 5 implies that the student's written response with regard to the use of mathematical symbolism was not developed to an adequate level. Firstly the equal to sign is not aligned to the equal to sign in the previous step. Further note that even if we accept this student's incorrect reasoning to solve the given system of equations his substitutions after the step $X=A^{-1}C$ are incorrect. This clearly indicates that the mathematical understanding of student S1 was instrumental in nature.

CONCLUSION

Mathematical understanding could be at an instrumental level (applying rules) or relational (knowing what to do and why). The use of mathematical symbolism could serve an instrumental role (to guide or keep track of one's thinking) and also a communicative function (to enable the sharing of problems and solutions). Since there were only two participants the findings of this study cannot be generalized. The framework arrived at to guide the examination of their written work served a useful purpose. It provided a structure for the researcher to get an insight into students' mathematical understanding, of the solution of systems of linear equations using matrix methods. The framework enabled the researcher to make conclusions on the mathematical understanding of those students with regard to: (1) the nature of their understanding displayed [instrumental or relational]; (2) their use of mathematical symbolism [instrumental role or communicative function]; (3) the degree to which their conceptual system was logically organized in terms of a schema or schemata.

With regard to the solving of a system of linear equations using matrix concepts, the following was noted for the first question: (1) Both students merely wrote down the augmented matrix related to the given system of equations and proceeded to use row operations. (2) Their written responses lacked a communicative function in that there was no evidence of them trying to communicate explicitly the method chosen or of their strategy to move towards a solution. (3)

The response of student S2 displayed that her understanding was at an instrumental level, with regard to her selection of row operations and also their execution with the intention of arriving at the row echelon form of a matrix. There was also evidence that she was unable to correctly interpret what she wrote, for example she simplified 5-2*a*+5-3*a* as -5*a*. (4) The overall response of student S2 revealed that she did not have a logically organized conceptual system for the method that she chose. (5) The overall response of student S1 revealed that her mental structures of actions, processes and objects were coherently linked to relevant schemata. Further there was evidence of suitable connections within such schemata. However, although she could execute mental structures, her execution was not developed to a level to externally communicate without hindrances to others. With regard to the use of an inverse matrix to solve a given system of equations it was found that both the students' understanding was at an instrumental level. In particular student S1's schema to solve a system of n linear equations in n unknowns was not sufficiently developed. It was found that the degree of interrelationships between schemata was lacking. In particular the schema for using the inverse of a matrix was not sufficiently connected to the relevant component of related matrix representation of an equivalent system, in the schema for solving systems of linear equations using matrix methods.

RECOMMENDATIONS

The first recommendation is that some sort of a framework should be used by lecturers to examine the written work of students with the focus on the level of mathematical understanding of their students. Such an examination of students' written work and relevant feedback to them could lead to an improvement in some aspects of their mathematical understanding, in particular the communicative function of mathematical symbolism. The second recommendation is that the framework to be used should be made available to students. Students should also be informed how the framework will be used to gauge their level of mathematical understanding.

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REFERENCES

Arnon I, Cottrill J, Dubinsky E, Oktac A, Fuentes SR, Trigueros M, Weller K 2014. *APOS Theory A Framework for Research and Curriculum Development in Mathematics Education.* New York: Springer.

Babajee DKR 2014. A Comparison between Cramer's Rule and a Proposed 2 × 2 Cramer-Elimination Method to Solve Systems of Three or More Linear Equations. From <https://www.researchgate.net/publication/255981900_ A_comparison_ between_ Cramer% 27s_ Rule_and_a_ proposed_2_by_2_ Cramer-Elimination_ Method_ to_solve_systems_of _three_ or_more_linear_equations> (Retrieved on 20 February 2015).

Barmby P, Harries T, Higgins S, Suggate J 2007. How Can We Assess Mathematical Understanding? From <http://www.emis.de/proceedings/PME31/2/41> (Retrieved on 6 August 2013).

Britton S, Henderson J 2009. Linear algebra revisited: An attempt to understand students' conceptual difficulties. *International Journal of Mathematical Education in Science and Technology*, 4(7): 963-974.

Dorier JL 1990. Continuous Analysis of One Year of Science Students' Work, in Linear Algebra, in First Year of French University. *Proceedings of the 14th Annual Conference for the Psychology of Mathematics Education*, Oaxtepex, Mexico, 15-20 July, II: 35-42.

Dorier JL, Sierpinska A 2001. Research into the teaching and learning of linear algebra. In: D Holton, M Artigue, U Krichgraber, J Hillel, M Niss, A Schoenfeld (Eds.): *The Teaching and Learning of Mathematics at University Level: An ICMI Study*. Dordrecht, Netherlands: Kluwer Academic Publishers, pp. 255-273.

Dubinsky E 2010. The APOS theory of Learning Mathematics: Pedagogical Applications and Results. *Paper presented at the Eighteenth Annual Meeting of the Southern African Association for Research in Mathematics, Science and Technology Education*. Durban, South Africa. 18-21 January.

Godino JD 1996. Mathematical Concepts, Their Meanings, and Understanding. From <http://www.ugr.es/ ~jgodino/articulos_ingles/meaning_ understanding. pdf> (Retrieved on 6 August 2013).

Hiebert J, Carpenter TP 1992. Learning and teaching with understanding. In: DA Grouws (Ed.): *Handbook of Research on Mathematics Teaching and Learning*. New York: Macmillan, pp. 65-97.

Lay DC 2006. *Linear Algebra and its Applications.* 3rd Edition. New York: Pearson.

Lazebnik F 1996. On Systems of Linear Diophantine Equations. From <http://www.math.udel.edu/~lazebnik/papers/dioph2.pdf> (Retrieved on 20 February 2015).

Maharaj A 2010. An APOS analysis of students' understanding of the concept of a limit of a function. *Pythagoras,* 71: 41-52.

Maharaj A 2013. An APOS analysis of natural science students' understanding of derivatives. *South African Journal of Education*, 33(1): 146-164.

Maharaj A 2014. An APOS analysis of natural science students' understanding of integration. *REDIMAT – Journal of Research in Mathematics Education,* 3(1): 53-72.

Maharaj A 2015. A framework to gauge mathematical understanding: A case study on linear algebra concepts. *International Journal of Educational Sciences*, 11(2): 144-153.

Ndlovu ZA, Brijlall D 2015. Pre-service teachers' mental constructions of concepts in matrix algebra: A South African case. *African Journal of Research in Mathematics, Science and Technology Education*, 19(2): 156-171.

Nickerson RS 1985. Understanding understanding. *American Journal of Education*, 93(2): 201-239.

Sierpinska A 1994. *Understanding in Mathematics*. London: The Falmer Press.

Sierpinska A, Nnadozie A, Okta A 2002. *A Study of Relationships between Theoretical Thinking and High Achievement in Linear Algebra*. Manuscript. Montreal, Quebec: Concordia University.

Skemp R 1976. Relational understanding and instrumental understanding. *Mathematics Teaching*, 77: 20-26.

Stewart S, Thomas M 2009. Linear Algebra Snapshots through APOS and Embodied, Symbolic and Formal Worlds of Mathematical Thinking. From <http://www.merga.net.au/documents/Stewart_RP09.pdf> (Retrieved on 11 November 2013).

Stylianides AJ, Stylianides GJ 2007. Learning Mathematics with Understanding: A Critical Consideration of the Learning Principle in the Principles and Standards for School Mathematics. From <http://www.math.umt.edu/tmme/vol4no1/TMMEv4n1a8.pdf> (Retrieved on 6 August 2013).

Tan ST 2005. *College Mathematics.* 6th Edition. United States: Brooks/Cole.

Tan ST 2014. *Applied Calculus for Managerial, Life and Social Sciences*. 9th Edition. United States: Brooks/Cole.

Vinner S 1991. The role of definitions in the teaching and learning of mathematics. In: DO Tall (Ed.): *Advanced Mathematical Thinking.* Dordrecht, Netherlands: Kluwer Academic Publishers, pp. 65–81.

Zaslavsky O, Shir K 2005. Students' conceptions of a mathematical definition. *Journal for Research in Mathematics Education*, 36(4): 317–346.

Weller K, Arnon I, Dubinsky E 2009. Pre-service teachers' understanding of the relation between a fraction or integer and its decimal expansion. *Canadian Journal of Science, Mathematics and Technology Education*, 9(1): 5-28.

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