© Kamla-Raj 2018
Int J Edu Sci, 21(1-3): 112-123 (2018)
DOI: 10.31901/24566322.2018/21.1-3.775

An Investigation into the Preparedness of Teachers to Teach Grade 12 Mathematics: A Case Study

Aneshkumar Maharaj

School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, South Africa
E-mail: maharaja32@ukzn.ac.za

KEYWORDS APOS Theory. Basic Knowledge. Skills. Teaching

ABSTRACT This paper reports on an investigation into the preparedness of a group of mathematics teachers in the Pinetown District, in KwaZulu-Natal, South Africa to teach learners grade twelve mathematics. The teachers were given tests in algebra, trigonometry and calculus. Those tests were based on common errors, misconceptions or difficulties relating to school mathematics made by university students and those reported in the examiners' reports for the national senior certificate examinations in mathematics. The written responses of those teachers were analysed. It was found that for this group of teachers there was a strong positive correlation between their errors, misconceptions or difficulties to those on which the tests were designed. An implication of this study is that one should not expect teachers to effectively communicate to those they teach, if they themselves do not have a good understanding of the content that they are expected to teach.

INTRODUCTION

In 2010 this researcher was approached by relevant stakeholders from the Electricity Supply Commission (ESKOM) in South Africa to facilitate a meeting with the then School of Mathematical Sciences at the University of KwaZulu-Natal (UKZN). At that meeting it was noted that they were concerned about the quality of graduates they worked with. They therefore wanted to work with relevant role-players in mathematics at university level to address this concern. Attendees at that meeting also included a mathematics representative from a local district of the Department of Education (DoE), members of the School of Mathematical Sciences whose research interest focused on the teaching of mathematics and a member of an NGO (Non-Government Organisation) who was a former mathematics subject advisor engaged with promoting the teaching and learning of mathematics at secondary school level.

The view expressed was that if suitable measures were put in place to adequately address the teaching and learning of mathematics then other things that are considered to be important will be taken care of. It emerged that ESKOM was concerned about the mathematical abilities of university students and graduates who seek employment with ESKOM and its' customers. The suggestion by the ESKOM representatives was that the focus should be on pooling together different initiatives to make an impact on the teaching and learning of mathematics. A number of interactive planning meetings were held to determine the research and project priorities. It emerged that to improve the teaching and learning of school mathematics there was a need to look at the problem at two levels:

1. At a research level which informs what needs to be done at the university and also in the schooling context, with the focus on improving the teaching and learning of mathematics. Research on what needs to be done to help grade 11 and 12 mathematics teachers and pupils was required.
2. Devising suitable support programs for mathematics teachers and learners. This included: (a) upgrading of teacher qualifications and teaching abilities; (b) the provision of quality interactive teaching material that could be used by teachers and pupils.

To realize these there was a need to set up a project at the Westville Campus of UKZN, the UKZN-ESKOM Mathematics Project, and to apply for the necessary funding for that project. A steering committee to direct what should happen at each of the two levels indicated above was assembled. The committee comprises of representatives from the School of Mathematical Sciences (now School of Mathematics, Statistics and Computer Science) at UKZN and the teacher education section of UKZN, ESKOM representatives, a Chief Education Specialist for Mathematics (DoE, KwaZulu-Natal) and an NGO representative. Members on the steering committee have considerable experience in the teaching and learning of mathematics at the school and post-school levels. For this committee to implement visions arrived at there was a need for the required funding. The Tertiary Education Support Programme (TESP) of ESKOM made available grants to fund the activities for focus that were conceptualised at the 2010 meeting.

Rationale for this Study

The preparedness of teachers to teach secondary school mathematics seems to be a worldwide issue. For example, the United States faces a significant shortage of well-prepared secondary mathematics teachers which has led to a serious concern on the quality of mathematics' instruction (Mathematics Teacher Education Partnership 2016). To address this concern those interested in the teaching and learning of mathematics should look at ideas to improve the mathematics education of teachers at their universities and communities (Zimba 2016). This was also the view at the meeting indicated in the previous sub-section that this researcher organised. In early 2011 after discussions with the Head of School of Mathematical Sciences, this researcher met with the DoE and NGO representative to discuss the planning of a pilot study, of a week's duration, with selected teachers from the Pinetown District. This was after the DoE representative indicated there was a need for such a programme in that district.

The grants awarded by ESKOM'S TESP were and are still used to address the problem at the two levels indicated above. Research teams worked on the following projects: Maths e-learning and Assessment project; upgrading of grade 12 mathematics teachers' knowledge and skills. The latter project focuses on teachers who wish to study level one university mathematics modules (from 2013), which count as credits towards their further studies at university, either as part of a BSc degree or the ACE (Advanced Certificate in Education). This paper focuses on the analysis of responses of a group of teachers to tests administered during a pilot study designed for upgrading grade 12 mathematics teachers' knowledge and skills relating to the content they are expected to teach. The theoretical framework for this paper is APOS (action-process-object-schema) mental constructions and this is how it differs from the paper by Brijlall and Maharaj (2014).

Objectives

The main objective was to answer, for the identified group of teachers, the research question: *What is the competency level of grade 12 practising teachers' mathematics content knowledge?* To answer this question it was decided to focus on the following: (1) What were some of the common errors made by grade 12 candidates in algebra, trigonometry and calculus? (2) What were the levels of teachers' knowledge relating to the basic content to be taught in these sections? (3) Is there a relationship between identified teachers' content knowledge and errors, misconceptions or difficulties to those of grade 12 candidates reported on in the mathematics examiners' reports?

Theoretical Framework

The theoretical framework for this study is APOS (action-process-object-schema) theory. APOS theory is based on the following assumptions (Dubinsky 2010; Maharaj 2010):

1. *Assumption on Mathematical Knowledge:* An individual's mathematical knowledge is his/her tendency to respond to perceived mathematical problem situations and their solutions by reflecting on them in a social context, and constructing or reconstructing mental structures to use in dealing with the situations.

2. *Hypothesis on Learning:* An individual does not learn mathematical concepts directly. He/she applies mental structures to make sense of a concept (Piaget 1964). Learning is facilitated if the individual possesses mental structures appropriate for a given mathematical concept. If appropriate mental structures are not present, then learning the concept is almost impossible.

If one accepts this then for teachers the above imply that the goal for teaching should consist of strategies for helping students build appropriate mental structures and guiding them to apply such structures to construct their understanding of mathematical concepts. However, this is only possible if the teacher in question has the relevant mental structures for the mathematical knowledge he or she is expected to teach.

The main mental mechanisms for building the mental structures of action, process, object, and schema are called *interiorisation* and *encapsulation* (Dubinsky 2010; Weller et al. 2003). The acronym APOS refers to the mental structures of action, process, object and schema. APOS theory postulates that a mathematical concept develops as one tries to transform existing physical or mental objects. The descriptions of action, process, object and schema given are based on those given by Weller et al. (2009) and Maharaj (2010, 2013, 2014, 2015).

1. Action: This is in the form of a transformation based on a reaction to stimuli which an individual perceives as external. The main characteristic is the need to perform each step of the transformation explicitly, resulting in physical external evidence of the performing of the action. It requires specific teaching. The successful execution of an action depends on an individual's level of relevant knowledge and skills required to perform the action.

2. Process: When an individual repeats and reflects on an action it could be *interiorised* into a mental process. So a process results in the development of a mental structure that allows an individual to perform wholly in the mind the same operation as the action. The main characteristic of a process is that the individual can imagine performing the transformation in the mind without having to execute each step explicitly.

3. Object: This occurs when the individual has *encapsulated* a process into a cognitive object. It requires that the individual becomes aware of a process as a totality and realises that transformations can act on that totality. The individual should be able to actually construct such transformations. The actual construction of such transformations to act on a totality could be explicit or in one's imagination.

4. Schema: A mathematical topic involves many actions, processes and objects. The construction of a schema requires that these need to be organised and linked by the individual into a *coherent framework*. An important characteristic of a schema is the emergence of a coherent framework that enables an individual to decide in the context of a particular mathematical situation, whether the schema applies or not.

Explanations offered by APOS analyses are limited to descriptions of the thinking which an individual might be capable of. In the case of teachers the implication is that their thinking (dependent on their mental structures) should impact on the effectiveness of their teaching. Such analyses of teacher solutions to problems they are expected to teach could be useful to determine limitations in their mental structures, which are reported in the section on *Results* and *Discussion* sections of this paper. Note however, that it is not asserted that such analyses describe what "really" happened in the teachers' minds, since this is probably unknowable.

Literature Review

A number of studies (Stacey 1988; Vinner 1991; Kieran 1992; Esty 1992; Sfard and Linchevski 1994; Bell 1995; Linchevski and Herscovics 1996; Souviney 1996; Dreyfus 1999; Lithner 2000; Mason 2000; Pyke 2003; Maharaj 2005, 2008; Brijlall and Maharaj 2014; Shuilleabhain 2015)) have focused on the teaching and learning of school mathematics. Those studies indicated important insights with regard to sources of students' difficulties in mathematics. Some of these are: (a) Mathematics makes use of a special language, symbolic notation, which fills a dual role as an instrument of communication and thought (Pyke 2003; Maharaj 2015). It is the use of symbolic notation that makes it possible to represent mathematical concepts, structures and relationships in symbolic form. (b) A student's inability to acquire an in-depth sense of the structural aspects of algebra could be the main obstacle (Kieran 1992). (c) The development of algebraic thinking is a sequence of ever more advanced transitions from operational (procedural) to structural outlooks (Sfard and Linchevski 1994). Algebraic thinking includes symbolic notation used to represent structures in trigonometry, for example $\frac{\tan\theta}{1+\sqrt{2}\cos\theta}$. (d) Learners have difficulty in recognising equivalent equations, interpreting the basic surface structure of equations, dealing with multi-term equations (including ones in which the unknown occurs on both sides) and decision-making with regard to which transformations are permissible and should be made in the context of the given equation (Maharaj 2008). (e) The "style and the nature of questions encountered by students strongly influences the sense that they make of the subject matter" (Mason 2000: 97). With regard to the latter, the questions that come to the mind of a teacher are influenced by the perspective and disposition that he/she has towards mathematics and pedagogy (Mason 2000; Maharaj 2008). (f) It is important for mathematics' teachers to correctly identify students' prior knowledge (Shuilleabhain 2015). This helps teachers to anticipate students' responses and to plan a relevant sequence of learning for their students. If these are accepted then a mathematics teacher's competency in the subject knowledge to be taught should influence the sense that his/her learners make of the subject matter.

METHODOLOGY

The participants were thirty two teachers who were identified by the Chief Education Specialist for Mathematics (DoE, KwaZulu-Natal) in the Pinetown region. That is the context of the case study referred to in the title of this paper. A letter of invitation to those teachers was sent by the School of Mathematical Sciences. The contents of that letter were finalized after consultations with the Head of School of Mathematical Sciences, the Chief Education Specialist for Mathematics (DoE) and the NGO representative. This included the topics for each day of the five day workshop.

The researcher then planned tests for (a) Algebra, (b) Trigonometry, and (c) Calculus. An email was sent to the module coordinators of first year university mathematics modules in May 2011. The email requested for feedback based on their marking of the first semester examination scripts with regard to basic mathematical knowledge and skills that they expected first year students to have acquired at school, but were lacking and impeded student success to study university mathematics. Items for those three tests were formulated after looking at the feedback received from our module coordinators and noting the findings of Maharaj (2005, 2008) and relevant national diagnostic reports (DoBE 2011, 2015). The relevant test, each of 20 minute duration, was administered to the participants before the relevant section began. Not all of the thirty two teachers participated in the tests. The reasons were that some: arrived after the test commenced; chose not to write a test or the tests; wrote the test but did not hand it in. For each of the three relevant days the written responses of participants were looked at by the facilitator and aspects that were found to be lacking were addressed. Each test was followed by a five hour workshop on the content and teaching of the relevant sections. However, the researcher decided to focus the analysis of data and discussion of findings for this paper only on the written responses of teachers to those three tests. The assumption for this was that the analyses and findings should give us an insight into what the actual competency of those grade 12 mathematics teachers was, before we made any interventions. To an extent this could be a picture of what the situation with regard to mathematics' teacher's subject knowledge competencies is likely to be in some of the other schools.

On the first day of the five day workshop each participant was supplied with a copy of a study guide (Maharaj et al. 2011a) which contained material on each of the grade 12 mathematics topics. The material included for each topic a summary of the basic content, examples, practice exercises, examination type questions with full solutions and topic tests. An answer booklet (Maharaj et al. 2011b) containing the answers to practice exercises and the topic tests was also given to each participant. Participants were encouraged to read on the relevant sections before coming to the workshop for the remaining days. They were informed that on the last day they would be given a test based on the topics for algebra, trigonometry and calculus. So for the separate tests on trigonometry and calculus this could have had an impact on those teachers' subject knowledge competencies. Also, the written consent of each participant was obtained to conduct research and to use data obtained for reporting purposes.

The researcher analysed the written responses of teachers to each of the three tests together with the comments made by the facilitators. For each of the three tests the results were summarised in table form. The coding for each of the Tables 1 to 3, indicated in the next section, was as follows: *Inadequate response* indicates a response that scored 50 percent or less of the mark for the question; *Response with some errors* indicates a response which was better than an inadequate response but not completely correct; *Completely correct response* indicates a response that was 100 percent mathematically correct.

RESULTS

These are presented under the following sub-headings: Algebra; Trigonometry; Calculus. The analysis of data for each test is given in Tables 1 to 3. In each of these tables, the columns "Section" and "Question type or structure" gives information on the type of question participates were exposed to. For example the third row in Table 1 implies that participants had to identify the type of equation as fully as possible given the equation structures:

$x(x\text{-}4) = \text{-}3$; $49\text{-}4(\text{-}4 - x) = 0$;
$x+8/x=6$ $x^2(x+2) = 5x+6$;
$5^{x+1} + 5^{1-x} = 26$.

Algebra

Table 1 indicates that fifteen respondents (75 percent) were able to express the number 42 as a product of its prime factors. A study of the written responses of the others indicated that 15 percent knew what was meant by the term factorisation, since they wrote 42=2x21. However, those teachers did not know what is meant by prime factors since they did not write 21 as a product of its prime factors. It could be concluded that in the context of APOS mental structures those teachers were at an action level. The responses of the remaining 10 percent indicated that they did not know what was required, which implied they did not know what the concepts *product*, *prime* and *factors* meant. This implies that those teachers had mental structures not even at an action level.

Observe that 50 percent of the teachers were *unable* to identify all five of the equations as fully as possible. This implies that they did not fully understand the dual role of symbolic notation, used to represent the equations, as an instrument of communication and thought (Pyke 2003; Maharaj 2015). In terms of APOS those teachers did not have satisfactory mental structures at the object level that enabled them to identify the given equations correctly.

Table 1 indicates that 50 percent of the respondents were unable to correctly solve the quadratic inequality, $x(x+4) > 5$ which implies that they did not have an adequate schema for the solving of a quadratic inequality. The solving of a quadratic inequality was indicated as an area of concern in the literature (Maharaj 2005; DoBE 2011). A recent examiners' report (DoBE 2015) noted that the solving of inequalities is still an area of concern.

It was observed that all respondents were able to correctly identify the functions $f(x) = -2x^2 + 4$; $g(x) = 3/x+1$ and $h(x) = 2^x$ (see Table 1). This suggests that they had adequate mental structures at the object level and an appropriate schema which they could use to identify those types of functions. However, note that only 25 percent of the teachers were able to correctly give the domain and range of all three functions. This implies that for 75 percent of the teachers the schema was not adequately developed or connected. Relating this to the literature review, DoBE (2011: 111) noted that functions "and graphs should be taught in a way that leads to an understanding of the effect of the different parameters". A study of the written responses

Table 1: Analysis of test results for algebra, including functions $(n = 20)$

Section	*Question type or structure*	*Inadequate response*	*Response with some errors*	*Completely correct response*
Prime factorisation	Write the number 42 as a product of its prime factors	2	3	15
Identifying the type of equation given as fully as possible	$x(x-4) = -3$ $49 - 4(-4 - x) = 0$ $x + \frac{8}{x} = 6$ $x^2(x+2) = 5x + 6$ $5^{x+1} + 5^{1-x} = 26$	8	2	10
Solving a quadratic inequality	$x(x+4) \geq 5$	8	2	10
Identifying given functions as linear, quadratic, logarithmic, exponential, hyperbole or cubic.	$f(x) = -2x^2 + 4$ $g(x) = \frac{3}{x+1}$ $h(x) = 2^x$	0	0	20
Domain and range of functions	Above functions	10	5	5

of those teachers (the 75%) in the sample indicated that for the concept of range, in the context of functions with such structures (Table 2 indicates they knew what was meant by range), those teachers were not even at a suitable action level. The reason for this conclusion is that about 70 percent of them had difficulty in determining the range of both the functions g(x)=3/(*x*+1) and h(x) = 2^x and . An observation of their written responses indicated illogical reasoning or there was no response. For example, illogical reasoning for the function was illustrated by the written response, 3=0.

Although 100 percent of the participants were able to identify the given three functions represented in symbolic form, only 50 percent of those participants were able to identify all five equations represented in symbolic form (see Table 1). Bearing in mind the importance the identification of the equation type serves as a prerequisite for the relevant algorithm to be applied, the implication is that more emphasis needs to be placed on identification of objects, specifically those symbolic structures representing equations, in the planning of teacher training and upgrading programmes.

Trigonometry

A perusal of the analysis of trigonometry results given in Table 2 indicates that in the main those teachers had the necessary knowledge and skills to answer the first three types of questions. For example, all of them were able to sketch the graphs of the two basic trigonometric functions for the given domain. About 80 percent of them were also able to correctly state the range of the function defined by y=sinθ. A study of the written responses which were coded as *responses with some errors*, indicated responses such as (-1,1) and (-1,1). This supports the emphasis on correct writing of intervals noted in DoBE (2011). What was surprising is that about two-thirds of those grade 12 mathematics teachers could not deduce the range of the next function , although most of them (about 80 percent) correctly stated the range of y-sinθ. Using APOS the implication here is that about two-thirds of those teachers did not have suitable mental constructions at the process level to determine the range of the next function.

Observe that most of those teachers displayed success in solving the trigonometric equation. However, the written responses of 50 percent of them were coded as *inadequate* for the following question: *For which value(s) of is the expression undefined?* With regard to the literature review this implies that their algebraic thinking was not suitably developed to enable them to make the leap to more advanced transitions, from operational (procedural) to structural outlooks (Sfard and Linchevski 1994). An examination of those teachers' written responses revealed that in the main there was a lack of strategy to determine when a mathematical expression is undefined. This also

Table 2: Analysis of test result for trigonometry (n=22)

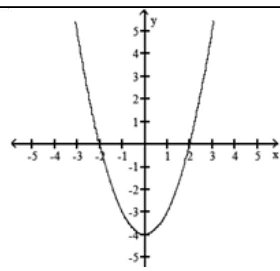
Section	*Type of question*	*Inadequate response*	*Response with some errors*	*Completely correct response*
Deductions from a given trig ratio	If $3\sin\theta = 1$, where θ is an acute angle, determine $\tan\theta$.	0	4	18
Sketching basic trig graphs	$y = \sin\theta,\ 0^\circ \le \theta \le 360^\circ$ $y = \cos\theta,\ 0^\circ \le \theta \le 360^\circ$	0	0	22
Range	$y = \sin\theta$	0	4	18
Deduction of range	$y = \sin^2\theta$	15	0	7
Solving trig equation	$2\cos^2\theta - 3\cos\theta = 2$, for $0^\circ \le \theta \le 360^\circ$	0	13	9
Undefined expressions	For which value(s) of θ is the expression $\frac{\tan\theta}{1+\sqrt{2}\cos\theta}$ undefined?	11	7	4

relates to finding the range of the function $g(x) = 3/(x+1)$ with which they had difficulty (see Table 1 and the observation). So once again their inadequate mental structures at the object level (for structures with a denominator) was exposed, and understandably the relevant schema for dealing with such structures was not adequately in place. Table 2 also indicates that about one-third of those teachers' responses to the question was coded as, *response with some errors*. The examination of such written responses indicated that most of those teachers didn't give complete general solutions for which the expression was defined, but specific values. For example, they correctly noted that: $1+2\cos\theta=0$, arrived at $\cos\theta=-1/\sqrt{2}$ and stated $\theta=150^0$ or $\theta=210^0$. Since the wording of the question did not place any restriction on , the correct deduction from $\cos\theta=-1/\sqrt{2}$ is: $\theta=150^0+\kappa 360^0$ or $\theta=210^0+\kappa 360$, where κ is an integer. With regard to a past national examiners' report the latter shortcoming was also detected in the responses of candidates (DoBE 2011: 112) which noted that for the general solution of a trigonometric equation "the learners should clearly understand that the angle is valid for any rotation".

Calculus

Observe that Table 3 indicates that the teachers were in the main successful in answering questions that required: (1) the application of the rules to find derivatives of the given types of functions, and (2) finding the equation of a tangent to a given curve at a particular value of x. Using APOS those teachers had mental constructions suitably developed at the action and process levels to answer these question types. Table 3 also indicates that more than half of those teachers did not know what information the derivative represented. Further 17 (about 70 percent) of them could not use the graph of a derivative to make conclusions with regard to where the original function (1) is increasing or decreasing, and (2) where its local maximum occurs. This is surprising since the information represented by the derivative would have been used by them to answer the question on the third section indicated in Table 3, to which about 90 percent of them gave a completely correct answer. These suggest that those teachers were good at ap-

Table 3: Analysis of test results for calculus ($n = 23$)

Section	*Type of question or structure*	*Inadequate response*	*Response with some errors*	*Completely correct response*
Information represented by $f'(x)$.	For a function $y=f(x)$ explain what information the derivative $f'(x)$ represents?	0	2	10
Applying rules for derivatives	$y=3x^2-4$ $f(x)=\dfrac{x^3-2x+1}{x}$ $h(x)=x(3x-4)$	1	1	21
Determine the equation of the tangent to the curve	$g(x)=-3x^2+2x+1$ at the point where $x=0$.	17	1	3
Interpretation of the graph of a derivative function	 The sketch shows the graph of $y=f'(x)$. Use the given sketch to determine each of the following: (a) Interval (s) over which function f is increasing. (b) Value (s) of x where the local maximum of f occur(s).			

plying rules and algorithms in calculus to answer routine questions, but did not have a deep understanding of the concept of a derivative.

DISCUSSION

This section focuses on insights from the national examiners' reports and the implications from the results section of this paper for the preparation of grade 12 mathematics' teachers to teach algebra, trigonometry and calculus.

Insights from Examiners' Reports

The study by Maharaj (2005) included an investigation of a number of Senior Certificate Mathematics Examiners' Reports which dealt with typical common errors, misconceptions or difficulties of candidates. It was found that many of those were still prevalent in the national examiners' reports of the Department of Basic Education (DoBE) for the period 2008 to 2011 (DoBE 2011) and for the period 2012 to 2015 (DoBE 2015). This begs the question*: Have we been running on the spot with regard to the training and upgrading of our senior secondary mathematics teachers?* Surely what appears in the mathematics examiners' reports should have impacted on the training or upgrading of the relevant teachers, and the situation would have improved.

Since the participants for this study were teachers who taught grade 12 mathematics during the period 2008 to 2011, the national diagnostic report (DoBE 2011) was initially focused on. This report indicated that for the period 2008 to 2011, for each of those four years, about 45 percent of the candidates who wrote the National Senior Certificate Examination for Mathematics achieved in the region of 30 percent to 39 percent. Further only about 30 percent of the candidates achieved percentages in the region of 40 percent and above. What this implies is that about 70 percent of the candidates for each of those four years achieved less than 40 percent in their mathematics examination. Further there has been a steady decrease in the number of mathematics candidates who wrote the examinations for each of the years from 2008 to 2011, and this "is a worrying trend" (DoBE 2011: 98). A possible reason for this could be that learners are encouraged to enrol for Mathematical Literacy instead of Mathematics, and the latter is more challenging to teach. This could be the situation at many schools since "a considerable number of centres are still producing candidates who do not understand basic subject content" (DoBE 2011: 98). The report (DoBE 2015) indicated that there was a general decline in the performance of candidates in the years 2014 and 2015. These reports (DoBE 2011, 2015) also noted that: (a) Many of the errors made by learners had their origins in a poor understanding of the basics and foundational competencies such as algebraic manipulation, factorisation, solution of equations and inequalities. (b) Interventions to improve learners' performance should focus also on knowledge, concepts and skills learnt earlier and not just on the final year of the grade 12. (c) Many candidates struggled with concepts in the curriculum that required deeper conceptual understanding, in particular questions which presented challenges were those that required candidates to interpret, explain or provide justification. (d) This suggests that many learners are exposed to 'stimulus-response' methods only, and as expected such learners will obviously have great difficulty when faced with questions testing the same procedures as previously but asked differently. In particular the reports indicated common errors, misconceptions and difficulties of learners relating to: (1) The solving of quadratic inequalities, for example, the type given in Table 1, where many learners were unable to write the correct intervals for the solution. (2) Questions which require using the graph to deduce properties of a function. (3) A lack of mastery of the work learnt in grades 9 and 10 prevents accurate answers even though grade 12 methods are understood. (4) The difference between an equation and an expression, for example -x+=0 and -x+1 respectively, so that valid operations for equations are incorrectly used for an expression, resulting in x-1. (5) Integrating knowledge of different sections to solve problems. Of concern is that it seemed learners were "mostly exposed to compartmentalised teaching and not an integrated approach, which is crucial in developing deeper understanding" (DoE 2011: 105). Further the reports (DoBE 2011, 2015) also noted that higher-order questions were very poorly answered and most candidates did not understand what was required. The question that arises is the following: *Could these be a result of the relevant mathematics teachers' lack of understanding of the basic subject content that they are expected to teach?* The *Re-*

sults section of this paper seems to indicate that a significant number of practising teacher participants lacked an adequate understanding of the subject content that they were expected to teach.

Algebra

The solving of the inequality, x(x+4)>5, requires applying algebraic manipulations to transform to an equivalent inequality x^2+x-5>0, recognizing that it is a quadratic inequality (an object) and then applying an appropriate algorithm to solve (the need for a suitable schema). The observation was that 50 percent of the respondents did not have suitable mental structures at the object level, for the solving of a quadratic inequality. Common errors detected from their scripts were: (a) $(x+5)(x-1) > 0 \Rightarrow x+5>0$ or $x-1>0$, (b) $(x+5)(x-1)>0 \Rightarrow x+5>0$ and $x-1>0$, and (c) $x< -5$ or $x> \Rightarrow 1< x < -5$. These are consistent with the literature (DoBE 2011: 87) which noted that "learners need to be taught different interval notations" and that the solving of inequalities is still an area of concern (DoBE 2015). It was observed that the learner common errors, misconceptions or difficulties related to interval notations could to some extent be traced to teacher common errors, misconceptions or difficulties relating to the subject content they are expected to teach.

There were serious gaps in about 50 percent of the participants' ability to solve the different types of equations that are dealt with in the grade 12 syllabus. This implies that such teachers would not be able to satisfactorily teach their learners relevant equation solving strategies when confronted with different types of equations. The reason for this conclusion is that an identification of the type of equation (an object) one is confronted with is a pre-requisite for the strategy or relevant algorithm required to solve the equation. For example, the equation $5^{x+1}+5^{1-x}=26$ needs to be rewritten as $5^x,5+5^1/5^x=26$ indicating that it is a *disguised exponential* quadratic equation which can be solved by using the substitution, $k=5^x$. An investigation into those teachers' written responses (the 50 percent under discussion) indicated that most of them (about 80%) could not fully identify this equation. Note that the algorithms for substitution technique, solving a quadratic equation and solving an exponential equation are required to solve the exponential equation, $5^{x+1}+5^{1-x}=26$. For this particular equation, one would require a suitable schema that enables one to connect the objects (different equation structures represented in symbolic form) and the relevant algorithms. Those teachers did not have such a schema. It is probable that such teachers, in their teaching, would not be in a position to help their learners to make relevant mental constructions with regard to objects and schema (Maharaj 2008). The implication here is that since they do not have such schema they would not be in a position to model the required thinking to solve such an equation. This could be a reason for many learners struggling "with concepts in the curriculum that required deeper conceptual understanding" (DoBE 2011: 99) and the answering of non-routine or unseen type questions (DoBE 2015).

Teacher training or upgrading should place more emphasis on the concept, range of a function, particularly in the context of such functions. For example this could include the effect of the parameters and in the context of the functions $G(x) = a/x+b$ or $H(x) = a(2)^{x+b}$. The emphasis should be on the effect these would have on the shape of graph of the function and on the domain and range in the context of the structure of the defining equation of the relevant function.

Trigonometry

Incorrect responses for the range of the function defined by $y=\sin\theta$ were (-1,1) and (-1,1). Having correctly found the range of function defined by $y=\sin\theta$ about 66 percent of those respondents were unable to deductively construct a process to determine the range of function defined by $y=\sin^2\theta$. These suggest that teacher training and upgrading programmes should focus on (1) the correct use of the round and square brackets in the context of interval notation, and (2) exposure to more problem solving situations that require deductive reasoning.

In the context of trigonometric expressions and equations teacher training or upgrading programmes should focus on: *When is a mathematical expression or equation undefined?* The answering of that question should include the context of trigonometric expressions and equations. The latter should include contexts were the domain is not restricted. Finally these should model an appropriate schema for dealing with such situations.

Calculus

In terms of APOS, for the concept of the derivative, the mental constructions of those teachers were not adequately developed at the object and schema levels. It could be argued that such teachers who do not have a deep understanding of a concept will be unable to plan and implement sequential lessons (Shuilleabhain 2015) that would lead their learners to a deep understanding of the relevant concept. With regard to the literature review the "style and the nature of questions encountered by students strongly influences the sense that they make of the subject matter" (Mason 2000: 97). The lack of understanding of what information the derivative represents was exposed when those teachers were confronted with the question indicated in the last row of Table 3. Such teachers would expose their learners to mostly "compartmentalise teaching and not an integrated approach, which is crucial in developing deeper understanding" (DoE 2011: 105). An integrated approach relies on a connected schema, which includes pulling together of symbolic notations and geometric connections.

The implication for teacher training or upgrading is that there should be a focus on the questions: (1) For a function *y=f(x)* explain what information the derivative *f'(x)* represents? (2) What information can be derived from the graphical representation of a derivative? (3) How can that information be organized into a format that leads to important user friendly information with regard to the characteristics of the original function? Since answers to these questions could lead to a deeper understanding of the derivative concept for teachers and learners, possible answers to these are provided.

(1) The derivative *f'(x)* gives the gradient of the tangent to the function *y=f(x)* at any point *(x,f(x))*on its graph. So *f'(x)* actually represents the gradient function of the original function *y=f(x)*. See Figure 1 for an illustration. This should be followed by a discussion of that happens to the gradient of the tangent, *f'(x)*, as the point of contact varies along the curve.

(2) The *derivative* is (a) *positive* where its graph is *above* the x-axis, (b) *zero* where it *intersects* or *touches* the x-axis, and (c) *negative* where its graph is *below* the x-axis.

(3) The researcher illustrate this using the question given in the last row of Table 3. Using (2) above we get the information on a number line, in Figure 2.

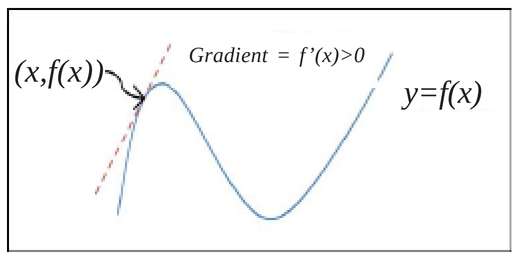


Fig. 1. f'(x) represents the gradient function of y=f(x).

Using Figure 2 the following conclusions can be made:

(a) The function *f* is increasing on the intervals (-∞,-2) or (2,∞)

(b) The function *f* has a local maximum at =-2.

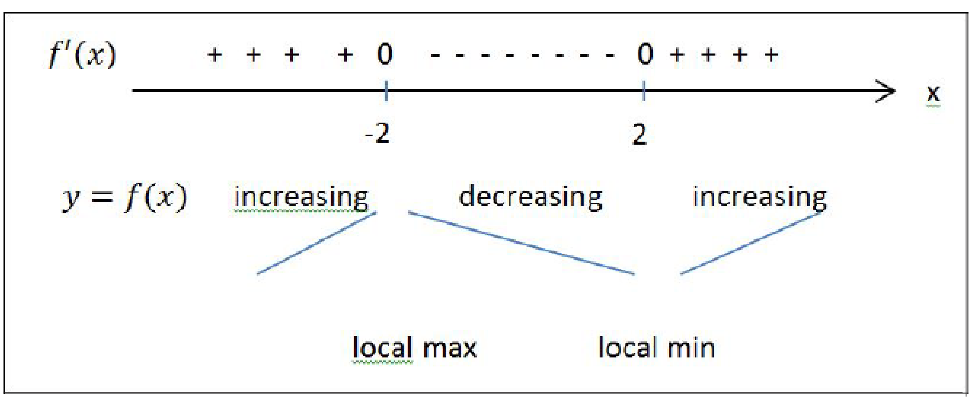


Fig. 2. Organization of information from the graph of the derivative

If the aim is to make problem-solving and non-routine, unseen questions an integral part of classroom teaching (DoBE 2015) then in-service and pre-service teacher training programmes need to address the identified problem areas and suggested implications that could improve the content levels of teachers. The aim of these programmes should be to facilitate the development of suitable schema that incorporate mental structures for the relevant APO (action-process-object) for the teaching of identified concepts or topics in algebra, trigonometry and calculus at grade 12 level. This could lead to teachers becoming more confident in the teaching of the content that they are expected to teach to their learners.

CONCLUSION

This study showed that for the sample there was a relationship between mathematics teachers' content knowledge and the learner errors, misconceptions or difficulties reported on in the mathematics examiners' reports. About 50 percent of the teachers in the sample group were unable to correctly solve a quadratic inequality of the type $x(x+4) > 5$. It was also evident the teachers needed more exposure to find the domain and range of functions which have the following types of structures $G(x) = a/x+b$ *or* $H(x) = a(2)^{x+b}$, which involve translations and reflections. This should also include vertical translations of functions. It was also found that over 50 percent of the teachers did not know what information is represented by the derivative of a function. Further, about 90 percent of the teachers could not make relevant deductions concerning the characteristics of the original function when the derivative of that function was given in graphical form. The *Results* and *Discussion* sections illustrated in the context of different content areas that the learner common errors, misconceptions or difficulties could be traced to teacher common errors, misconceptions or difficulties relating to the subject content they are expected to teach.

RECOMMENDATIONS

It was apparent that many of the teachers in this study did not have the mental constructions at the appropriate object and schema levels for the mathematics subject content they were expected to teach. If such a situation is allowed to persist then this would continue the trend of their learners being exposed to mostly compartmentalised teaching, lacking appropriate integration and deep understanding of the mathematics involved. The *Results* and *Discussion* sections of this paper give insights on what needs to be done to rectify the situation that possibly exists in many of our schools in South Africa. It is recommended that teacher training institutions and those responsible for teacher upgrading also focus on the development of higher order mental structures, in particular at the object and schema levels in the context of APOS theory, for the concepts and topics indicated in the *Results and Discussion* sections of this paper.

ACKNOWLEDGEMENTS

This study was made possible by grants received from ESKOM's Tertiary Education Support Programme (TESP) for the *UKZN-ESKOM Mathematics Project.*

REFERENCES

Bell A 1995. Purpose in school algebra. *Journal of Mathematical Behaviour*, 14: 41-73.

Brijlall D, Maharaj A 2014. Exploring support strategies for high school mathematics teachers from underachieving schools. *International Journal of Educational Sciences*, 7(1): 99-107.

Dreyfus T 1999. Why Johnny can't prove. *Educational Studies in Mathematics*, 38: 85-109.

Department of Basic Education 2011. *Report on the National Senior Certificate Examination 2011: National Diagnostic Report on Learner Performance*. Pretoria: Department of Basic Education.

Department of Basic Education 2015. *Report on the National Senior Certificate Examination 2015: National Diagnostic Report on Learner Performance*. Pretoria: Department of Basic Education.

Dubinsky E 2010. The APOS Theory of Learning Mathematics: Pedagogical Applications and Results. *Paper presented at the Eighteenth Annual Meeting of the Southern African Association for Research in Mathematics, Science and Technology Education*. 18 - 21 January, University of KwaZulu-Natal, Durban, South Africa.

Esty WW 1992. Language concepts of mathematics. *Focus on Learning Problems in Mathematics*, 14: 31-54.

Kieran C 1992. The Learning and Teaching of School Algebra. In: DA Grouws (Ed.): *Handbook of Research on Mathematics Teaching and Learning*. New York: Macmillan Publishing Company, pp. 390-419.

Linchevski L, Herscovics N 1996. Crossing the cognitive gap between arithmetic and algebra: Operating on the unknown in the context of equations. *Educational Studies in Mathematics*, 30: 39-65.

Lithner J 2000. Mathematical reasoning and familiar procedures. *International Journal of Mathematical Education in Science and Technology*, 31: 83-95.

Maharaj A 2005. *Investigating the Senior Certificate Mathematics Examination in South Africa: Implications for Teaching*. PhD Thesis, Unpublished. Pretoria: University of South Africa.

Maharaj A 2008. Some insights from research literature for teaching and learning mathematics. *South African Journal of Education*, 28: 401-414.

Maharaj A 2010. An APOS analysis of students' understanding of the concept of a limit of a function. *Pythagoras*, 71: 41-52.

Maharaj A 2013. An APOS analysis of natural science students' understanding of derivatives. *South African Journal of Education*, 33(1): 146-164.

Maharaj A 2014. An APOS analysis of natural science students' understanding of integration. *REDIMAT*, 3(1): 54-73.

Maharaj A 2015. A framework to gauge mathematical understanding: A case study on linear algebra concepts. *International Journal of Educational Sciences*, 11(2): 144-153.

Maharaj A, Reddy D, Mogamberry M 2011a. *Mathematics Grade 12 (NCS).* Revised Edition 2011. Durban: TESS Publishing.

Maharaj A, Reddy D, Mogamberry M 2011b. *Mathematics Grade 12 (NCS) Answer Booklet*. First Edition 2011. Durban: TESS Publishing.

Mason J 2000. Asking mathematical questions mathematically. *International Journal of Mathematical Education in Science and Technology*, 31: 97-111.

Mathematics Teacher Education Partnership 2016. Update: Transforming Secondary Mathematics Teacher Preparation. From <http://www.aplu.org/projects-and-initiatives/stem-education/SMTI_Library/mte-p-update-2016/file> (Retrieved on 12 September 2016).

Piaget J 1964. Development and learning. *Journal of Research in Science Teaching*, 2: 176-180.

Pyke CL 2003. The use of symbols, words, and diagrams as indicators of mathematical cognition: A causal model. *Journal for Research in Mathematics Education*, 34: 406-432.

Sfard A, Linchevski L 1994. The gains and the pitfalls of reification — the case of algebra. *Educational Studies in Mathematics*, 26: 191-228.

Shuilleabhain AN 2015. Developing Mathematics Teachers' Pedagogical Content Knowledge through Iterative Cycles of Lesson Study. From <https://hal.archives-ouvertes.fr/hal-01289593/document> (Retrieved on 12 September 2016).

Souviney RJ 1996. *Learning to Teach Mathematics*. 2nd Edition. New York: Macmillan Publishing Company.

Stacey K 1988. A framework for the analysis of mathematical problem solving behaviour. *Journal of Mathematical Behaviour*, 6: 353-360.

Vinner S 1991. The role of definitions in the teaching and learning of mathematics. In: D Tall (Ed.): *Advanced Mathematical Thinking*. London: Kluwer Academic Publishers, p. 65-81.

Weller K, Clark J, Dubinsky E, Loch S, McDonald M, Merkovsky R 2003. Student performance and attitudes in courses based on APOS theory and the ACE teaching cycle. In: A Selden, E Dubinsky, G Harel, F Hitt (Eds.): *Research in Collegiate Mathematics Education V*. Providence, Rhode Island: American Mathematical Society, pp. 97-131.

Weller K, Arnon I, Dubinsky E 2009. Pre-service teachers' understanding of the relation between a fraction or integer and its decimal expansion. *Canadian Journal of Science, Mathematics and Technology Education*, 9(1): 5-28.

Zimba J 2016. The Common Core and the Potential for Mathematicians to Improve the Teaching of School Mathematics. From <http://www.ams.org/publications/journals/notices/201602/rnoti-p154.pdf> (Retrieved on 12 September 2016).

Paper received for publication on October 2015
Paper accepted for publication on October 2016