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An Investigation into Mathematics Major Student Teachers' Patterns of Errors on Grade 12 Mathematics at a School of Education
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KEYWORDS epistemic obstacles, algebra, functions

ABSTRACT This study investigated first year mathematics major students' errors patterns on grade 12 mathematics particularly on the topics of algebra and functions. Prominent errors that the students still hold on high school mathematics tasks were identified. The study also aimed to explain underlying causes of students' errors. The research focussed on a sample of (n=63), 2012 first year Bachelor of Education (BEd)mathematics major students at a School of Education. The students wrote mathematics tasks selected from past National Senior Certificate Examinations. Their scripts were analysed to detect common error patterns. A smaller sample of ten students was selected for in-depth interview on the basis of errors evidenced in their scripts. In audio taped interviews, students were probed to manifest justifications for their errors. The research found out that most errors students had were conceptual. Students also showed that in the main they did not understand mathematical notation and terminology meaning that they did not even understand what the tasks required of them. The interviews suggested that at times, students hold alternative mathematical conceptions which they cannot fully defend under scrutiny. At times they showed that they just did not have the knowledge required to solve tasks.

The research recommends that error analysis be central to a bridging grade 12 mathematics course that must be launched to all first year to BEd mathematics major students. This course must use different representations and models to develop students' conceptual understanding identified as the main source of errors in this report.

INTRODUCTION

This study was undertaken at a School of Education in Johannesburg, South Africa on mathematics major students to determine these students' error patterns on Grade 12 mathematics. The research was done by two lecturers who teach in the Division of Mathematics Education at the school. The research is practice-based in that it identifies problems in our workspace and seeks to understand them in order to improve our practice. Practice based research in education is centred on 'evidence-based' policy and practice (Ros and Vermeulen 2010). Practice based or practitioner led research is a form of applied research. The Organisation for Economic Co-operation and Development (OECD) (2002) define applied research as 'original investigation undertaken in order to acquire new knowledge.., directed towards a specific practical aim or objective' (p.30). In an educational setting, practice based research aims to identify problems in educational practice, gather evidence on it and interpret the evidence in order to improve teaching and learning. Such research may lead to real change in education as it fills the gap between theory and practice. According to Ros and Vermeulen (2010), practice based research is more helpful to effecting change in teaching and learning than many theoretically based research papers which tend to be too academic and difficult to read and access. Practitioners find it difficult to access and apply such theory based research in their teaching. Also such articles tend to have little practical recommendations.

Practised based research starts with problems in a specific context (Ros and Vermeulen 2010). It is meaningful to schools; and colleagues can collaborate in doing the research as well as announcing the results to local staff. Therefore practice based research is likely to have more local impact. Ros and Vermeulen (2010) argue that practice based research has five characteristics. Firstly it has to be pragmatic in that it solves current real-world problems. Secondly it is grounded in both theory and the real-world context. Thirdly it is interactive, iterative and flexible. Fourthly, it involves collaboration among researchers and practitioners in a nonlinear process. Lastly it is contextual and lead to contextually-sensitive design principles and theories.

Statement of the problem

Error analysis and diagnosis is a critical aspect of pedagogy. Schulman (1986) argued that knowledge of students' errors and misconception in a discipline is a critical part Pedagogical Content Knowledge (PCK). PCK is knowledge for teaching of a particular subject that is quite different from the subject matter itself or pedagogic knowledge per ser. It concerns knowing ways of formulating and presenting subject matter so that it is comprehensible to learners. Knowledge of learner errors and misconceptions is particularly important because when students learn new mathematics concepts, they often invent alternative concepts quite different from the ones teachers expect them to learn (Nesher, 1987; Smith, diSessa & Roschelle, 1993). This happens because of learners' failure in the adaptation process as they assimilate new knowledge into current schemas in a situation which actually calls for the reorganisation of the old schema or its complete replacement through accommodation. Knowledge of students' alternative conceptions or misconceptions is important for teachers' PCK as it helps to devise strategies for learners to revise their concepts.

We argued in this study that if students' erratic mathematical thinking and reasoning is not unpacked for both teachers and learners' consideration, little progress in learning mathematics can be expected, because teaching may not be addressing learners' current difficulties. As learners' misconceptions are identified and resolved one by one, the learning of new mathematics concepts becomes easier for them. This is because mathematics is an hierarchical subject in which higher mathematical concepts can only be understood if lower ones have been also been understood. Also, research suggests that a single misconception, a bug, can snowball to reproduce a cluster of errors along the way (Nesher, 1987). The snowballing effect of misconceptions makes learning mathematics hugely difficult. A misconception effectively locks out the understanding new concepts, whereas understanding a concept ushers understanding of new concepts. In mathematics the success of understanding one concept leads to the success of understanding another and vice versa. Therefore knowledge of learners' errors associated with varied mathematical tasks is imperative for the progression of mathematics teaching and learning including in teacher education.

In South Africa research on learner error analysis has been quite minimal. Although regular international standardised tests involving South Africa; (see for example Howie 2001; Moloi and Chetty 2011; Taylor and Taylor 2013) focussed on performance in mathematics (and science), the results were mainly aimed at national performance comparisons. Very little data on the errors learners hold is discussed. However of late, some researchers are beginning to take this research field quite seriously (see for example; Luneta and Makonye 2012; Makonye and Luneta 2014). Luneta in particular regards error analysis research in mathematics (and physics) as imperative it could leverage teachers' effectiveness to take learners' understanding of these subjects to a level not yet reached in the South African educational landscape given the documented low performance of South African learners in mathematics and science.

This research aims to understand the pattern of errors and misconceptions that first year Wits School of Education mathematics major students have on grade 12 mathematics on the topics of functions and algebra.

The research questions are:-

(a) What is the pattern of errors shown by student teachers in responses to Grade 12 algebra and functions Grade 12 mathematics tasks?

(b) What explanation can be availed for these patterns?

Significance

In our school we have embarked to improve our teaching through practice based research.. We do this under the theme "we research what we teach; and teach what we research". Over the years we have realised the wide and compelling mathematics knowledge deficits of our student teachers. We have noted that most of our students arrive at university without the core mathematics competencies that they are supposed to have acquired at high school. Also during students’ teaching experience, these gaps in mathematics content knowledge often crop in the delivery of mathematics lessons. Such ignorance is embarrassing to the students themselves and the learners they teach not to mention their university. In some schools, supervising teachers complain that our student teachers’ content knowledge hardly measures up. Given this scenario, it is imperative that we address this problem so that the teachers whom we train graduate with impeccable mathematics teaching qualifications. A research such as this one which focuses student teachers’ Grade 12 mathematics knowledge is important. It is important in that once the research is done, students’ weaknesses can addressed by the time they graduate.

It is also our experience that some teachers even graduate from South African universities with gaps in Grade 12 level mathematics. Such teachers are not assets but liabilities who perpetuate sub-standard levels of mathematics teaching and learning in this country. In that event society have the right to point fingers to us the mathematics educators as having failed them.

Theoretical framework and conceptual framework

First, we draw from Davis’ (1984) ideas that liken the working of learners’ minds to computer information processing systems. According to Davis the human mind systematically operates according certain ways. Davis advocated that knowledge representation in a learners’ mind is known as a frame. One of the laws that govern frames is the Brown-Matz-van Lehn Law (Davis, 1984) which stipulates that top level programs must run to their ultimate conclusions. In this case a learner has an overarching, very important procedure that governs the solution of a mathematics task. This higher procedure may require lower procedures to provide inputs into it for it to operate to its logical end. If these lower procedures are not available, accommodations are made, so that default procedures can be constructed so that the higher level programs do not stop but can be completed. The default input programs may be erratic. So in the super-sub procedure interface, when obstacles are met, modifications are made to enable processing to continue. Davis asserts that the super or sub-procedures invariably have a valid mathematical basis where there have been used productively in the past but may be wrongly retrieved in situations where they are not called for. This results in errors in learners’ productions.

Davis (1984) also maintained the Feigenbaum Minimal-Discrimination Rule (FMDR); which states that information processing systems will make essential discriminations but may not make discriminations finer than necessary, that is to say discriminations are not made where there are presently not needed. As an example, he illustrated the primary-grade undifferentiated binary-operation frame. He explained that since learners first learn addition at school, when learners are asked use other operations in their school lives, they ignore other operation signs and always add. They are fixated to the addition operation, the operation they first encountered at school. Davis claimed that frame retrieval may be cued by brief, explicit, specific cues or situational similarity. The frames, micro worlds and scripts can be compared to concept images (Tall & Vinner, 1981).

Davis reported on the characteristics of frames. Firstly, they serve as assimilation schemas for organising input data. Secondly, he argued that errors reveal the inner workings and functioning of frames. Thirdly, frames all have correct earlier learning. Fourthly that each frame demands certain input information to be provided, if not available default information is inputted. Davis argued that the persistence of frames is evidenced by the fact that frames had identical functioning in a variety of situations. Frames also seem to have a hold on people using them, and people go to great extends to protect their cherished frames. This is evidenced by individuals being prepared to go to bizarre distortions to avoid challenges to the content of their frames. Davis also pointed that any mismatches to the frame results in modification of input data rather than the frame itself. To mathematics educators, Davis has no good news as he asserted that "instruction often avails little against frames" (p. 123). He documented that that "even if instruction produces a change in frame, this change is often not permanent; before long the frame reasserts itself and behaviour reverts back to what it was before instruction" (p. 125). Piaget (1968) highlighted that the reversion to earlier wrong frames was due cognitive developmental immaturation.

Davis explained that the creation and operation of frames follows orderly rules, rule of initial overgeneralisation, rule of the FMD rule; that is discriminations are not made where they are not presently needed, top level programs must run and contradictory semantic information (Davis & McKnight, 1980) is frequently not influential in modifying algorithmic behaviour.

Our framework further presumes that errors and misconceptions can be understood in terms of imperfectly constructed concept images. Tall & Vinner (1981, p.152) describe the *concept image a*s “the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes activated at a particular time when the concept image is evoked”. They contrast the concept image to the *concept definition* which is the expression in words of the concept as it is formally held by the mathematical community. We presume that errors and misconceptions in mathematics occur if learners’ concept images are at variance with concept definitions. Errors and misconceptions are aggravated and perpetuated if the learner’s attempt to incorporate new concepts into inadequate or defective concept images.

In reconciling these frameworks, we maintain that learners have errors and misconceptions as they refer new mathematical concepts to past conceptual frames they are not prepared to let go. Their misconceptions are also due to concept images that are the sum of the old frames and the new mathematical contexts they encounter. The frames are bolstered by the FMDR and the Brown-Matz-van Lehn Law that compels them to complete procedures even when if they use inputs they are not sure of. In their justified maintenance of their conceptual frames and concept images, they fail to learn mathematics as they always look back instead of forwards to the concept definitions provided by their teachers and other learning resources. Errors and misconceptions are aggravated and perpetuated if learners incorporate new knowledge into inadequate and defective concept images.

Davis’ theory can be linked to Tall & Vinner’s concept image and concept definition. Essentially the higher and lower procedures of working out mathematical tasks that Davis refers to can be compared to Tall & Vinner’s in the sense that when the learner acquires the concept which is aligned to the appropriate concept image that he has, the learner will operate at the higher procedure mode. Davis’ frames seem to be static whereas concept images are dynamic.

Conceptual framework

These errors are based from in Nolting (1997)’s work.

Type of error	Description
Conceptual error	Conceptual errors are due to student's failure to reason or operate in settings inv careful application of concept definitions, relations, or representations. Examples; $0.214 > 0.3$; $3 + 2x = 5x$; $f(x + 2) = f(x) + f(2)$; $\frac{d}{dx}[(3x + 1)^2] = [\frac{d}{dx}(3x + 1)]^2$
Procedural error	Failure to use an algorithms, these are steps to follow a rule to obtain a correct a . Example: $\begin{array}{rcc} & T & U \\ & 1 & 6 \\ + & & 2 \\ \hline \end{array}$ giving answers of 9; from $1 + 6 + 2$ or 38; from $6 + 2 = 8$ and $1 + 2$ $= 3$ since 1 has no pair so 2 becomes the default evaluation.
Application error	These occur when one has sufficient conceptual or procedural knowledge but fail to apply them to solve a task Example: $x + x = x^2$
Careless error	These are errors of performance due to sloppiness or tiredness or some other interference. They differ from the above three errors which are errors of competency.

Student's errors were analysed with this conceptual framework. We next discuss the methodology.

Methodology

This exploratory on-going study randomly selected the students with the expectation of gaining in-depth and sufficient information about prospective teachers' patterns of misconceptions and errors displayed when working on school mathematics tasks. The sixty three (63) male and female students were drawn from year 1 prospective mathematics specialists in the BEd mathematics programme. The data collection techniques utilized were document analysis and interviews. The preliminary analysis discussed in this paper draws on data from the analysis of prospective teachers responses to algebra and functions tasks adapted from the 2011 Grade 12 National Senior Certificate paper 1 and 2.

These two papers were distributed to the mathematics prospective teachers. No examination restrictions were applied when the students wrote the papers. The researchers intended to solicit as much data as possible and as such students were allowed to complete the papers at their own leisure. A return rate of 58% was realized. The papers covered the five broad areas of the mathematics curriculum, namely; algebra, functions, sequences and series, trigonometry and analytical geometry. Mathematics questions vary with respect to content and the cognitive processes requires in finding solutions to the questions. The implication is that different cognitive demands are likely to induce different kinds of responses. We employed the Stein, Smith, Henningsen and Silver (2000) mathematics task analysis framework to analyse the cognitive demand of the questions. The questions were classified as low level questions categorized as procedure without connections according to the Stein et al. (2000) Mathematical Task Analysis Guide. The tasks required use of procedures requiring no explanations and limited cognitive demand for successful completion. Ten students were selected to probe them for their error patterns and misconceptions in answering the functions and algebra questions.

Results and discussions

We present the data analysis for item 1 (algebra) and item 5 (functions) of paper 2. The items covered the following areas:

	Question	Topic
item 1	Solve for *x*, correct to TWO decimal places, where necessary: 1.1.1 $x(x-1)=12$	Equations
	1.1.2 $7x^2+18x-9>0$	Inequalities

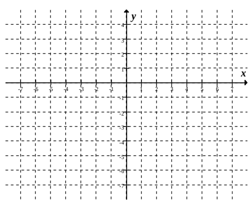
	1.1.3 Simplify completely, without the use of a calculator: $\left(\sqrt[5]{\sqrt{35}+\sqrt{3}}\right)\left(\sqrt[5]{\sqrt{35}-\sqrt{3}}\right)$	Surds
item 5	Consider the function $f(x)=\dfrac{3}{x-1}-2$.	
	5.1Write down the equations of the asymptotes of f.	Equation of asymptote
	5.2 Calculate the intercepts of the graph of f with the axes.	Intercepts
	5.3 Sketch the graph of f on the grid below. 	Sketching graphs
	5.4 Write down the range of $y = -f(x)$.	Range of a function

Table 1: Items content

Content analysis was utilized to examine the responses on the scripts. Three lenses were employed to analyse the data; a broad analysis of attempt and non-attempt of items, analysis of type of responses, and analysis of misconceptions and errors. We identified two broad categories of response and non-response. The response category refers to an attempt to answer the item whereas non-response referred to no attempt on the item. Table 2 shows the students' attempts and non-attempts of the items using percentage frequencies.

		Attempts (%)	Non-attempts (%)
item 1 (algebra)	1.1.1	100	0
	1.1.2	87	13

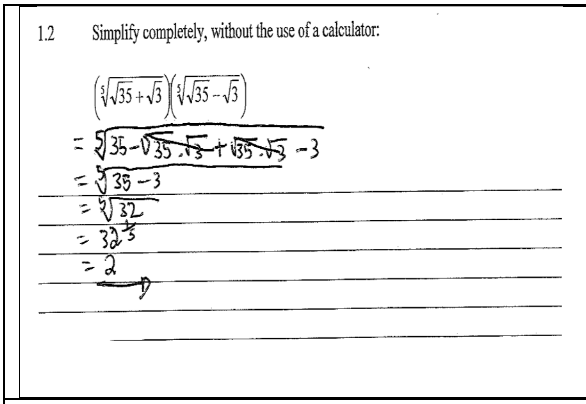
	1.1.3	61	29
item 5 (functions)	5.1	65	35
	5.2	61	29
	5.3	57	43
	5.4	70	30

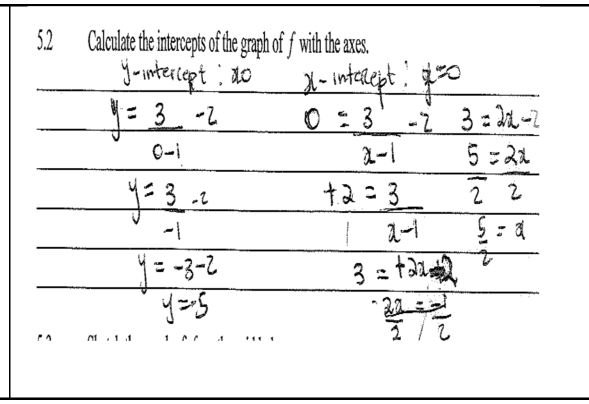
Table 2: Analysis of responses and non-responses

The table reflects that item 1.1.1 was attempted by all the students compared to item 1.1.3 which was attempted by 61% of the students indicating that 29% did not attempt the question. All the 63% students attempted the equation item but 13% failed to attempt the question on inequalities as much as they did not attempt the question on surds. Item 5 comprised on questions on functions. At least 60% of the students attempted the items although that cannot be said for item 5.3 where 43% did not attempt to sketch the hyperbola with 70% giving the range of the function. Overall more responses were realized than non-responses.

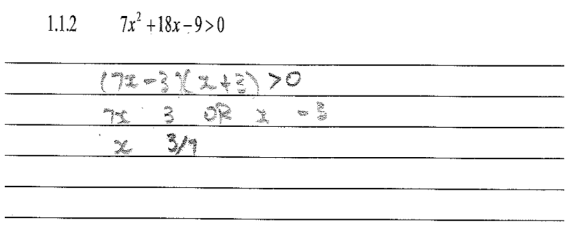
A deeper examination was conducted the reveal the nature of the response of the attempted items (see Table 3 below). Categories of responses were developed and characterized as, correct, partial, and incorrect. The correct response indicated that the participant's solution was correct. The strategy for the correct method was not considered at this stage. Partial responses were identified as those responses that were incomplete regardless of whether correct or incorrect. The partial responses had some elements of correct procedure that could have lead to an accurate response. Incorrect responses were identified as incorrect statements that resulted with incorrect solutions. Below are examples of responses according to the categories:

Correct responses

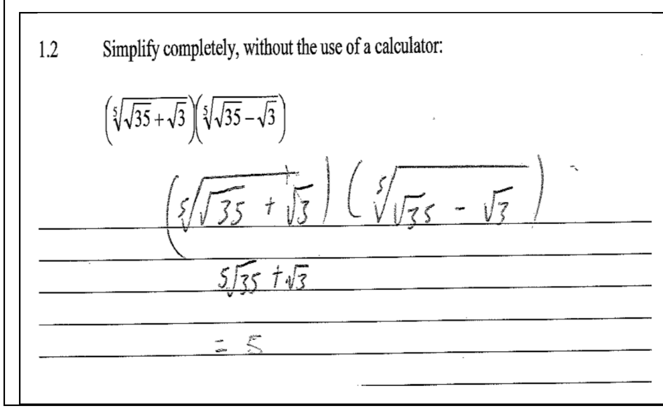




Partial responses



Incorrect responses



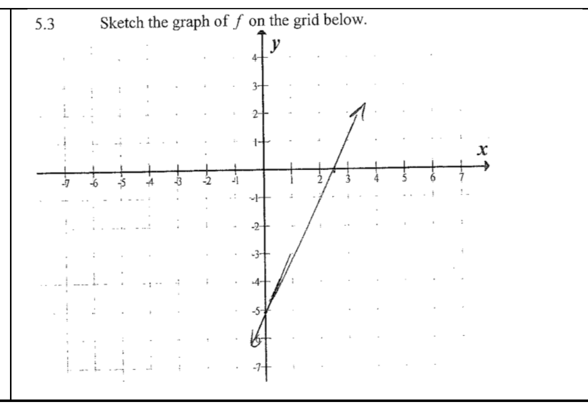


Figure 1: Type of responses to tasks

The focus of the investigation was to identify the types of errors and misconceptions displayed by the students. A further examination of the responses was crucial in identifying the errors and misconceptions. The categories of the responses provided a picture of the nature of responses (see Table 3).

		categories			
		Correct (%)	Partial (%)	Incorrect (%)	Non-response (%)
Item 1	1.1.1	70	13	17	0
	1.1.2	17	49	21	13
	1.1.3	17	0	44	39
Item 5	5.1	57	4	4	35
	5.2	31	26	4	39
	5.3	14	4	39	43
	5.4	9	4	57	30

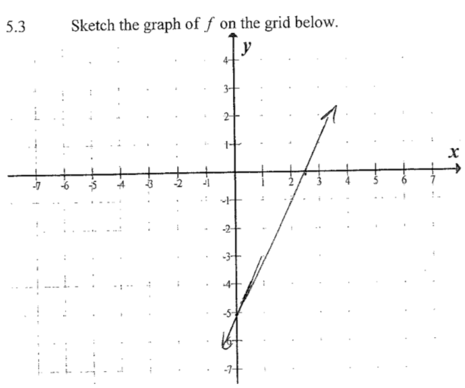
Table 3: analysis of item 1 and 5 responses

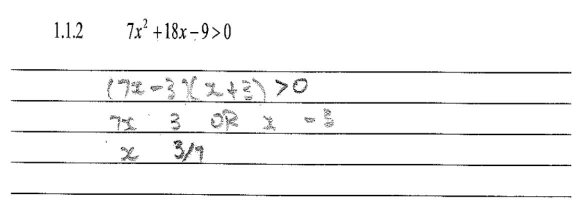
For instance, although item 1.1.1 was attempted by all the students, 70% correctly answered the item with 17% incorrect and 13% partial responses. Item 5.4 was attempted by 70% with 9% of these students successfully giving the range of the function and 57% unable to do. Item 1.1.2 was attempted by 87% of the students but only 17% correctly answered the item. We attribute this to lack of understanding of the use of the inequality sign. The students failed to establish the meaning of the inequality symbols suggesting that they could not create meanings for rules and procedures that govern actions on the inequality symbol. One student expressed that;

I don't have problems solving equations but then… then with inequalities, I can solve the left side then …. ahh… I don't know what to do with the sign.

It is a practice in South Africa that inequalities are taught as a subordinate subject of equations. Boero and Bazzini (2004) suggest that this approach "trivializes" inequalities, resulting with emphasis "on routine procedures, which are not easy for students to understand, interpret and control" (p.140). The students performed poorly on inequalities, surds, sketching hyperbolic function and finding the range of such functions.

We intended to identify the errors and misconceptions displayed in responding to the mathematics paper. To do this we realized that errors could be located among the partial and incorrect responses. The types of errors were characterized as, conceptual, procedural, application and unintended errors using the classification by Nolting (1998). While many authors have provided several classifications of errors (see for example, Donaldson, 1963; Hirst, 2003), we preferred the Nolting (1998)'s categories of errors. According to Nolting (1998), conceptual errors are those made when learners do not understand the mathematical properties or principles required to successful answer the tasks. Procedural errors are made due to learners' misuse of a rule, formula or algorithm when carrying out a mathematical calculation. An application error occurs when a learner understands the concepts but cannot apply them to a specific problem situation. A careless or unintended error occurs when the learners have the required knowledge to perform a task correctly but due to some interference or distraction, the learner makes errors. However, we included another category, incomplete response with correct statements. Below (fig. 2) are examples of responses according to the categories errors produced:

<table>
<tr>
<td>Conceptual errors


</td>
<td>Application errors


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<tr>
<td>Procedural errors</td>
<td>Unintended errors</td>
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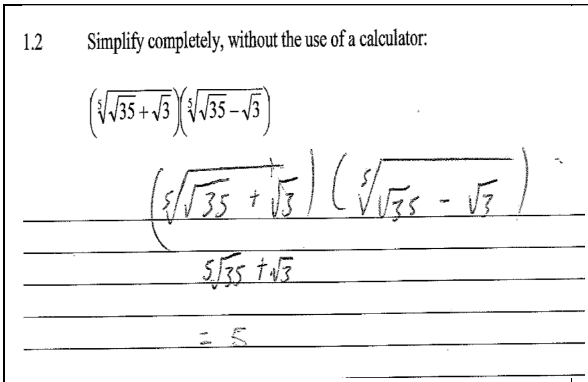

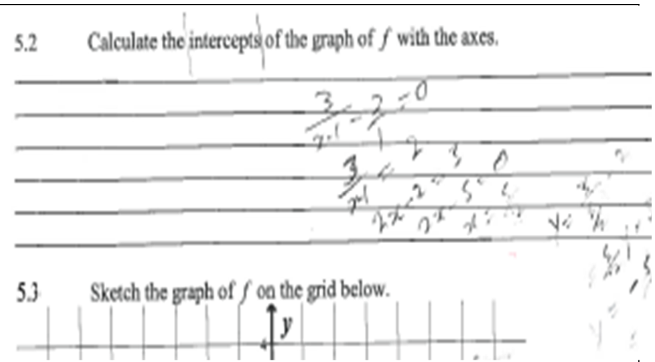

Incomplete correct errors

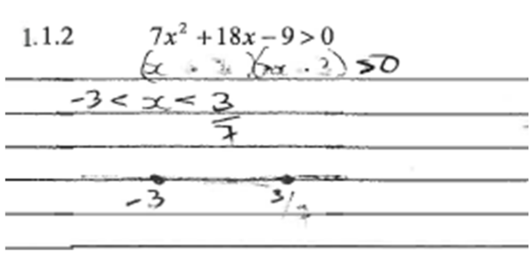

Figure 2: Exemplar errors

Table 4 shows the summary of error classifications for the partial and incorrect responses for each item.

		errors				
		Conceptual	Application	procedural	unintended	Correct/ incomplete
Item 1	1.1.1	28	14	14	44	0
	1.1.2	6	31	25	19	19
	1.2	30	30	30	10	0
Item 5	5.1	50	0	0	50	0
	5.2	28	0	14	28	28
	5.3	50	40	0	10	0
	5.4	58	21	0	7	14

Table 4: Analysis of errors

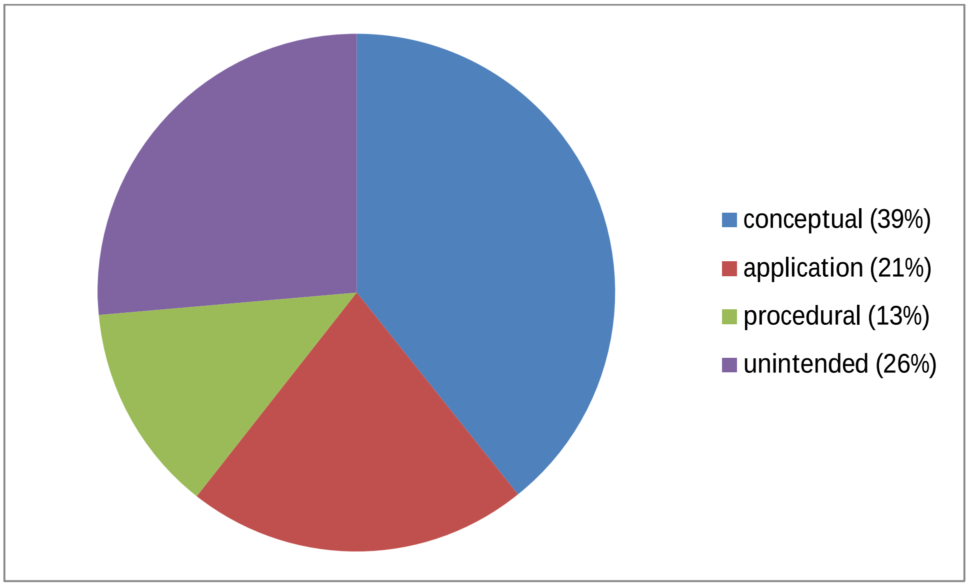

Figure 3: Types of errors committed in algebra and functions items

Fig. 3 shows that conceptual errors (39%) are more prominent than other errors. There is an indication that students do not understand the mathematical properties or principles required to successfully answer the tasks. This is particularly clear that more conceptual errors were realised in responding to the functions task. Of great interest is that application errors correlate with procedural errors. The application and procedural errors students had were due to weak competence in algebraic processes such as the notions of a variable and substitution. Students also had errors due to mathematical symbolism and terminology. Difficulty with the conceptualisation of the symbolic representation impacts on the understanding of the function concept. It indicates that the students who did not possess sufficient knowledge of the function concept were bound to make application errors. What is depicted by Question 5.3 is that the students could not shift from one functional representation to the other. Mathematical connections between concepts are induced when different systems of representation are promoted. This is line with Duval's (1999) suggestion that mathematics activities should provide opportunity for "the construction of a cognitive structure by which the students can recognize the same object through different representations” (p.12). The task required the students to relate the visualization process through the transfer of symbolic objects by processing and representing these objects to a visual representation. Clearly, this process requires a transformation of a perceptual apprehension of the symbols to a sequential apprehension of the concepts. Although 57% correctly identified the asymptote (question 5.1) only 14% could sketch the graph (question 5.3), suggesting that they were bound to make conceptual errors and application errors. As mentioned, unintended error occurred when the learners have the required knowledge to perform a task correctly but due to some interference or distraction, the learner makes errors. The greatest unintended errors occurred for questions that had the highest correct response rate.

We note that students have concept images (Tall and Vinner) of algebra and functions which are different from what they have been taught. In their way to construct mathematical concepts they often resort to cognitive representation systems (Davis, 1984) inadequate for the considerably high level mathematics that was required in these tasks. Students tried to assimilate new mathematical ideas into present weak cognitive structures they had instead of building new ones. The patterns of errors displayed in this paper are a cummulation of students’ cognitive problems with school mathematics over many years, that remain unaddressed. The students have carried misunderstanding of mathematics ideas and procedures over many years culminating in fuzzy mathematical ideas they showed in this study. The lack of conceptual understanding effectively blocks students’ epistemic access and progression in mathematics at all levels.

Conclusion

This research identified patterns of errors student teachers made in their responses to various mathematics problems basing on an argument that errors an indication of some misconception. A misconception indicates a misunderstanding of a concept and an error indicates a misapplication of a concept. The research found out that most errors students had were due to weak competence in algebraic processes including the notions of a variable and substitution. Students also had many errors due to mathematical symbolism and terminology. The findings indicate that conceptual errors were more prominent than the application, procedural and unintended errors, clearly suggesting that the mathematical proficiency level was low. Although the students know how to use the procedures they could still not apply these procedures to specific situation and thus the occurrence of a substantive number of application errors.

Recommendations

In view of this research on the patterns of errors of student teachers on algebra and functions tasks, we recommend that:

- all grade 12 mathematics be re-taught for the first semester in year 1 of the mathematics major students Bachelor of Education course. This re-teaching must take into consideration the need to strengthen conceptual understanding through using multiple representations and models.
- in the implementation of the course, lecturers take cognisance of the errors and misconceptions students have so that teaching is directed at these epistemic obstacles.

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