International Journal of Education Sciences (IJES)
Received on 02. 07. 2014/03. 11. 2014
REF. NO. IJES-539
(Always refer our Reference No. for all correspondence)
An Outline of Possible in-Course Diagnostics for Derivatives and Integrals of Functions
Aneshkumar Maharaj
School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, South Africa
E-mail: maharaja32@ukzn.ac.za

An Outline of Possible in-Course Diagnostics for Derivatives and Integrals of Functions

Aneshkumar Maharaj

School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, South Africa
E-mail: maharaja32@ukzn.ac.za

KEYWORDS outcomes, diagnostics, derivatives of functions, integral of a function

ABSTRACT In this paper we documented our findings on the formulation of expected student learning outcomes for derivatives and integrals, in the context of calculus. We used our literature review and the expected student learning outcomes to document sample diagnostic questions for the sections on derivatives and integrals that should facilitate student learning of concepts in these sections.

INTRODUCTION

At the University of KwaZulu-Natal relatively low pass rates for the first year differential calculus (Math130) module was of concern. Previous papers of ours (in pre-print) looked at the pre-course diagnostics for differential calculus and in-course diagnostics for functions, elementary logic, and limit and continuity of a function. In this paper our focus is to outline an in-course diagnostics of the section on derivatives and integrals of functions, the latter for basic functions. This aims at assessing the strengths and shortcomings of technical knowledge and skills, of the student. The diagnostic tests would not be for grading students but rather to provide feedback on their strengths and weaknesses with regard to content and skills on functions. We provide detailed learning outcomes for the topic on derivatives and some integrals for the first year differential calculus course, as offered at the University of KwaZulu-Natal.

The research questions for this paper were: *What are the expected learning outcomes with regard to the sections on derivatives and integrals of functions? How could we develop in-course diagnostics on derivatives and integrals for differential calculus as offered at our university?*

REVIEW OF LITERATURE

Our focus here was on diagnostic testing, derivatives (including the chain rule) and integral of a function.

Diagnostic testing

Our papers, *An outline of possible pre-course diagnostics for differential calculus* and *An outline of possible in-course diagnostics for functions* both in pre-print, discuss in detail the rationale for diagnostic testing and how one should go about formulating diagnostic questions. For the convenience of the reader we summarise the main points: (a) Clear learning outcomes for sections should be documented and these should be made public (Council of Regional Accrediting Commissions, 2004). (b) The formulation of the diagnostic questions should be guided by the relevant expected learning outcomes (Adam, 2006). (c) Pre-course paper based or computer based diagnostic testing has been and continues to be used by many institutions in the United Kingdom (Learning and Technology Support Network Maths Team Project, 2003). (d) There is evidence that diagnostic testing has led to improvement of student performance (Betts, Hahn & Zau, 2011). (e) Diagnostic tests could help a student if feedback is given relating to the strengths and weaknesses of the student, which could help him or her plan and take remedial measures to attend to identified weaknesses (The California State University, 2012). Our literature review indicated that many institutions used diagnostic testing to gauge the readiness of students to study calculus. In this paper our focus was on the formulation of expected learning outcomes and diagnostic questions for in-course diagnostics relating to derivatives and antiderivatives of functions. In our opinion these could improve the performance of students studying these, in particular at the University of KwaZulu-Natal and generally in developing countries.

Derivatives of functions, including the chain rule

There are a number of studies on students' understanding of the concept of a derivative of a function. For details on some of those studies the reader could refer to Maharaj (2013). Some of the important points are summarised for the reader: (a) The derivative is a difficult concept for many students to understand (Orton, 1983; Uygur & Özdas, 2005). (b) A commonly used *description* of the derivative is the following: gradient of a function $f(x)$ at x_0 is the slope of the tangent line to the curve f at the point $(x_0, f(x_0))$. Further, one should be careful when distinguishing between a *description* of a concept (which specifies some properties of that concept) and the formal concept *definition* (Giraldo, Carvalho & Tall, 2003). (c) Students' understanding of the derivative can be improved if they are exposed to several different kinds of representations, to process the derivative (Hähkiöniemi, 2004). For example this could be done by exposing students to different perceptual (eg. rate of change of the function from the graph, steepness of tangent) and symbolic (eg. differentiation rule, slope of a tangent) representations of the concept of a derivative. (d) Roorda, Vos & Goedhart (2009) found that growth in understanding depends on a variety of connections, both between and within representations, and also between a physical application and mathematical representations. So this implies that there should be a focus on representations and their relevant connections, as part of understanding derivatives. (e) It seems that students prefer the graphical representation in tasks and explanations about derivatives (Zandieh, 2000). This was also noted by Tall (2010) who made a strong argument for direct links between visualization and symbolization when teaching the concept of a derivative. (f) If the function considered is a composite function, then students' difficulties with the derivative increase and get worse (Tall, 1993). This could be the reason for the chain rule being one of the hardest ideas to convey to students in calculus (Gordon, 2005, Uygur & Özdas, 2007). The difficulties with the chain rule and its' applications, for a large number of students, could be attributed to their difficulties in dealing with composition and decomposition of functions (Clark, Cordero, Cottrill, Czarnocha, DeVries, St. John, Tolias & Vidakovic, 1997). So, the understanding of composition of functions is an integral part to understanding the chain rule (Webster, 1978; Cottrill 1999; Horvoth, 2007).

Antiderivative, integral of a function

A number of studies (eg. Abdul-Rahman, 2005; Ubuz, 1993; Orton, 1983; Sevimli & Delice, 2010; Haciomeroglu, Aspinwall & Presmeg, 2009; Maharaj 2014) have focused on student understanding of integration. The following conclusions could be made from those studies. (a) Differentiation can be viewed as a *forward* process and the difficulties faced by students in this concept are not as complicated as those in the reverse or *backward* process of integration (Abdul-Rahman, 2005). This implies that teaching needs to relate the antiderivative concept with that of the derivative. For example $\int f(x)dx$ represents the general antiderivative of $f(x)$ so $\int f(x)dx = F(x) + C$ provided $F'(x) = f(x)$. (b) Integration has a dual nature. It is both the inverse process of differentiation and a tool for calculation, for example area. For example with regard to this inverse process the derivative of a^{kx} is $k \ln a . a^{kx}$, therefore an antiderivative of a^{kx} is $\frac{1}{k \ln a} a^{kx}$ so $\int a^{kx} dx = \frac{1}{k \ln a} a^{kx} + C$. (c) Orton (1983) suggested that student errors with regard to the concept of integration could arise from their failure to appreciate the relationships involved in the problem or to their grasping of a principle essential to the solution, or failure to take account of the constraints laid down in what was given, or failure to carry out manipulations (though the principle involved may have been understood). Further, some errors could involve elements of more than one of these. (d) Kiat (2005) in a study found students had difficulty with questions on integration of trigonometric functions and also applying integration to evaluate plane areas. The students generally lacked both conceptual and procedural understanding of integration. The errors committed were technical errors, primarily attributed to the students' lack of specific mathematical content knowledge. (e) Visualization in the graphical context could help students to understand the relations between differentiation and integration (Ubuz, 1993). (f) A student's use of area under a curve is helpful in problem solving only when a deeper understanding of the structure behind the definite integral is present (Sealey, 2006). (g) There should be a focus on the relationship between the graphical and symbolic integral representations. Sevimli & Delice (2010) argued that this could increase the performance of solving definite integral problems. (h) Students' understanding can be enriched by changing thinking processes and establishing reversible relations between graphs of functions and their derivative or definite integral graphs (Haciomeroglu, Aspinwall, & Presmeg, 2009).

The implication of the above is that a variety of representations should be used and students should be encouraged to engage with a flexibility of mathematical conceptions (Andresen, 2007; Maharaj, 2010) of the derivative of the function f and the its integral, $\int f(x)dx$.

CONCEPTUAL FRAMEWORK

This was guided by the literature review and the following principles:

1. There is a conceptual hierarchy in the body of mathematics.

2. It is important for the learning outcomes for the unit or module, to be clearly documented. Further, students should know explicitly at the outset the learning outcomes expected of them.

3. For effective learning to occur it is not good enough for an instructor (teacher, lecturer or tutor) to be aware of the technical knowledge outcomes of a course/module. The documented learning outcomes should guide the formulation of suitable diagnostic questions, for students.

4. When students attempt the diagnostic questions there should be provisions for remedial activity, to overcome their identified shortcomings.

METHODOLOGY

The literature review, conceptual framework and study of the aims and content for the differential calculus (Math130) module informed the methodology. The researcher looked at the aim and content as indicated in the handbook of the Faculty of Science and Agriculture (2010), which is in the public domain. These are indicated below, the parts in italics are my emphasis:

Aim: To introduce and develop the Differential Calculus as well as the fundamentals of proof technique and rudimentary logic.

Content: Fundamental Concepts - elementary logic, proof techniques. Differential Calculus - Functions, graphs and inverse functions, limits and continuity, *the derivative, techniques of differentiation, applications of derivatives, antiderivatives.*

We then used our experience relating to teaching at both secondary and tertiary education institutions to formulate and document:

1. In-course expected learning outcomes for the sections on *the derivative of a function, techniques of differentiation, applications of derivatives, antiderivatives (integrals) of some standard functions*. We formulated such outcomes for the Math130 module by studying its aim and content (as indicated above), and also past assessment questions.

2. In-course diagnostics for *the derivative, techniques of differentiation, applications of derivatives, antiderivatives.* We used the learning outcomes identified to formulate questions on course content for the derivative of a function, techniques of differentiation, applications of derivatives, and antiderivatives or integrals of standard functions.

FINDINGS AND DISCUSSION

These are presented in the following order: (a) The derivative of a function, and (b) Antiderivatives and integrals. For each of these the learning outcomes are stated and then the sample diagnostic questions we formulated are given.

The derivative of a function

First the expected learning outcomes that were formulated are presented, followed by the sample diagnostic questions. The sample diagnostic questions are indicated under the following subheadings: derivatives from first principles, techniques for differentiation, and applications of derivatives.

Learning outcomes

We expect a student to be able to:

1. recall the defining condition for the existence of the derivative of a function at a given point or a given domain;
2. recall the defining condition for the existence of the derivative of a function on the whole domain;
3. determine if a given function has a derivative at a given point in its domain or on the entire domain;
4. compute the derivatives of standard functions from first principles;
5. deduce and recall the laws of derivatives;
6. recall the derivatives of standard functions;
7. compute the derivatives of new functions;
8. recognize standard observations emerging from the derivative of a function. Includes comparison of the graphs of a function and that of its derivative

- derivative is constant on an interval
- derivative is increasing or decreasing on an interval
- derivative is zero at a point
- derivative changes sign
- derivative is a multiple of the reciprocal of the variable
- derivative is an exponential
- derivative is periodic.

Diagnostic questions

This concerns defining conditions for a derivative of a function to exist at points in its domain and on the whole domain, laws of derivatives, derivatives of standard and new functions, observations emerging from the derivative of a function and their applications to standard and new functions.

Sample questions on derivatives from first principles

Past experience indicated that students confuse situations where the calculation of the derivative is required from first principles. Informed by the relevant learning outcomes that we documented the diagnostic questions arrived at are indicated in Table 1. In framing those questions we drew from our literature review on the importance of the formal concept *definition* (Giraldo, Carvalho & Tall, 2003).

Table 1: Derivatives from first principles

No.	Question	Answer
1.	State the defining condition for the derivative of a function $f(x)$ to exist at $x = a$ in its domain.	Condition: $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists.
2.	State the defining condition for the derivative of a function $f(x)$ to exist on its whole domain.	Condition: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists for each value of x in its domain.
3.	For the function $g(x) = \sqrt{x}$ determine from <u>first principles</u> the derivative, $g'(x)$, and state the values for which it exists.	$g'(x) = \frac{1}{2\sqrt{x}}$ Domain of $g(x)$ is $[0, \infty)$. $g'(x)$ exists for all x in $(0, \infty)$.

Sample questions on techniques of differentiation

Using the relevant expected learning outcomes we arrived at diagnostic questions on the rules for differentiation of new functions, including exponential and logarithmic functions. A sample of those questions is given in Table 2. Since the chain rule is one of the hardest ideas to convey to students in calculus (Gordon, 2005; Uygur & Özdas, 2007) we formulated a number of questions based on the application of this rule, see question 6. Those questions require the application of the chain in the context of different mathematical structural representations. We felt that student growth in understanding the techniques for differentiation could be promoted by exposing them to the application of the chain rule in the context of various mathematical representations (Roorda, Vos & Goedhart, 2009). A study of those questions indicate that the application of the chain rule is embedded in mathematical structures requiring for example, first applying the power rule or the rule quotient rule for differentiation [see questions 6a) and 6b) respectively].

Table 2: Differentiation of new functions, including exponential and logarithmic

4.	If $f(x)$ and $g(x)$ are functions defined on their domains, complete the follows laws for finding derivatives of functions: a) For some constant k, $y = k \Rightarrow y' =$ ___. b) For some constant $k, y = kf(x) \Rightarrow y' =$ ___ c) $y = f(x) \pm g(x) \Rightarrow y' =$ _____ d) $y = f(x).g(x) \Rightarrow y' =$ _______. e) $y = \frac{f(x)}{g(x)} \Rightarrow y' =$ _________ f) $y = f(g(x)) \Rightarrow \frac{dy}{dx} =$ ______	a) 0 b) $kf'(x)$ c) $f'(x) \pm g'(x)$ d) $f'(x).g(x) + f(x).g'(x)$ e) $\frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$ f) $f'(g(x)).g'(x)$
5.	Complete by stating the derivative of the following standard functions: a) $f(x) - x^n$ for $n \neq 0$ b) For $n \neq 0, y = [g(x)]^n$ c) $h(x) = e^{kx}$, k a non-zero constant. d) $y = a^{kx}$, k a non-zero constant and $a > 0$ $but\ a \neq 1$. e) $y = \ln(kx)$, k a non $-$ zero constant f) $y = \log_a(kx)$, k a non-zero constant	a) $f'(x) = nx^{n-1}$, $n \neq 0$ b) $y' = n[g(x)]^{n-1}g'(x)$ c) $h'(x) = ke^{kx}$ d) $h'(x) = k \ln a . a^{kx}$ e) $y' = \frac{1}{x}$ f) $y' = \frac{1}{(\ln a)x}$

	and $a > 0$ but $a \neq 1$.	
6.	Find the derivatives of the following functions: a) $y = 4x^2 - x + 3$ b) $f(x) = \sqrt{3x^4 + 1}$ c) $h(x) = \frac{6x - 2}{3x^2 - 2x}$ d) $g(x) = 3e^{4x} + (5x - 1)^e$ e) $y = \ln(5x^2 + x)$ f) $y = (3)^{-x} + \log(1 - x)$ g) $y = (x^2 - x)^{\frac{e}{2}}(1 - 3x)^{100}$	a) $y' = 8x - 1$ b) $f'(x) - \frac{6x^3}{\sqrt{3x^4 + 1}}$ c) $h'(x) = \frac{-18x^2 + 12x - 4}{(3x^2 - 2x)^2}$ d) $g'(x) = 12e^{4x} + 5e(5x - 1)^{e-}$ e) $y' = \frac{10x+1}{5x^2+x}$ f) $y' = -\ln 3\,.(3)^{-x} - \frac{1}{\ln 10(1-x)}$ g) $y' = -300(x^2 - x)^{\frac{e}{2}}(1 - 3x)^{99}$

We formulated a number of questions based on differentiation of trigonometric and other functions (see Table 3). Since differentiation and integration are reverse processes we noted the findings of Kiat (2005) who in a study found students had difficulty with questions on integration of trigonometric functions. Our feeling was that one needs to first focus on the differentiation of trigonometric functions, and later make the relevant links with integration. Further note that we once again focused on many different mathematical structures which required the application of the chain rule (see questions 8 and 9). We felt that it was important for students to analyse the mathematical structure given, first detect from the overall structure and then the embedded structure the rule or rules for differentiation to be applied (see question 8). Once this type of mathematical cognition is attained then they could concentrate on the applications of the relevant techniques for differentiation (see question 9).

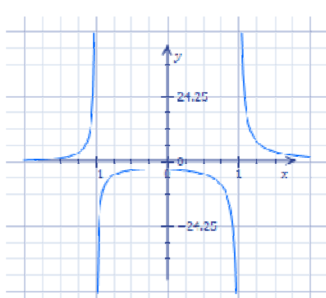
Table 3: Differentiation of trigonometric and other functions

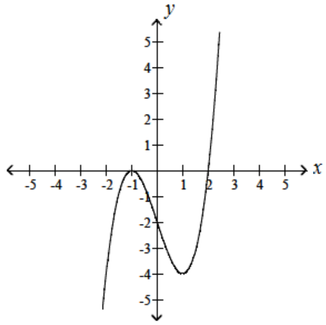
7.	For each of the following trigonometric function find the derivative $\frac{dy}{dx}$ if k is a non-zero constant. a) $y = \sin(kx)$	a) $k\cos(kx)$ b) $-k\sin(kx)$ c) $k(\sec(kx))^2$ d) $-k\csc(kx)\cot(kx)$ e) $k\sec(kx)\tan(kx)$

	b) $y=\cos(kx)$ c) $y=\tan(kx)$ d) $y=\csc(kx)$ e) $y=\sec(kx)$ f) $y=\cot(kx)$	f) $-k\,(\csc(k\,x))^2$
8.	Which rules will you use to compute the derivatives of the following functions with respect to x: a) $\cos^3 x$ b) $3x^7\tan(2x)+4$ c) $\sqrt[5]{\sec(\sin x)}$ d) $\frac{1}{\sqrt{x^2-x}}$ e) $\frac{(x^5-x+1)^{234}}{(x^6+x-2)^{567}}$ f) $\frac{x^2}{4}+y^2=1$	a) Chain rule b) Addition rule, product rule and chain rule c) Chain rule d) Laws of indices and chain rule e) Quotient rule and chain rule f) Implicit differentiation
9.	Find the derivatives of the function in question 8 above.	a) $-3\ \sin(x)\ (\cos(x))^2$ b) $6\,x^7\,(\sec(2\,x))^2+21\,x^6\,\tan(2\,x)$ c) $\frac{1}{5}(\sec(\sin x))^{-\frac{4}{5}}\sec(\sin x)\tan(\sin x)\cos x$ *or* $\frac{\cos(x)\sin(\sin(x))}{5\,(\cos(\sin(x)))^{\frac{6}{5}}}$ d) $-\frac{2x-1}{2\,(x^2-x)^{\frac{3}{2}}}$ e) $\frac{234(x^5-x+1)^{233}(5x^4-1)(x^6+x-2)^{567}-567(x^6+x-2)^{566}(6x^5}{(x^6+x-2)^{1134}}$ f) $y'=-\frac{x}{4y}$

The questions that we formulated for applications of the derivative are indicated in Table 4. We arrived at those question types from the expected learning outcomes that we documented. We were also informed by the important points that we noted in our literature review with regard to students' understanding of the derivative concept. Some of those points were: (a) Students' understanding of the derivative can be improved if they are exposed to several different kinds of representations (Hähkiöniemi, 2004), see questions 11, 12 and 13. (b) Roorda, Vos & Goedhart (2009) found that growth in understanding depends on a variety of connections, both between and within representations. The questions in Table 4 were designed to provide a variety of connections between and within representations. (c) It seems that students prefer the graphical representation in tasks and explanations about derivatives (Zandieh, 2000; Tall, 2010). That was one of the reasons for giving the graphical representation and the defining equation of the rational function in question 11. Another was the argument by Mahir (2010) that learning about functions and their graphical interpretation with suitable connections could ensure that students effectively learn concepts in calculus. We formulated question 12 to facilitate the development of such connections in the context of calculus concepts.

Table 4: Applications of derivatives

10.	What is meant by point of inflection of a curve?	A point on the curve where the concavity of the curve changes.
11.	Study the graphical illustration for a portion of the graph defined by the equation $y = \frac{x+3}{x^2-1}$. For this function determine the y-intercept, the defining equations of the asymptotes, the intervals over which the graph is increasing or decreasing, the intervals of concavity, and the local extrema.  	y-intercept is -3; vertical asymptotes are defined by $x = -1,\ or\ x = 1$, horizontal asymptote x-axis defined by $y = 0$; increasing over $(-\infty, -1) \cup (-1,0)$, decreasing over $(0,1) \cup (1, \infty)$; concave up over $(-\infty, -1) \cup (1, \infty)$. concave down over $(-1,1)$; local maximum of -3 at $x = 0$.
12.	The sketch shows the graph of the function $y = h'(x)$. Interpret the sketch and use it to answer the following questions.	a) A relative minimum occurs at $x = 2$. b) $-1;\ 1$

	 a) Identify the extrema of the function h. b) Identify the values of x where the points of inflection of curve of h occur. c) Discuss the concavity of the graph of h.	c) concave up on $(-\infty, -1) \cup (1, \infty)$ and concave down on $(-1, 1)$
13.	The function $M(x) = -\frac{1}{45}x^2 + 2x - 20;\ 30 \le x \le 65;$ is an approximation for the number of kilometres per litre of fuel used by a new prototype car, when driven at a speed of x kilometers per hour. a) Calculate the speed intervals over which the number of kilometres per litre of fuel increases or decreases. b) Calculate the absolute maximum number of kilometres per litre and the speed at which this is achieved.	a) increases on the speed interval $(30, 45)$ and decreases on the speed interval $(45, 65)$ b) Absolute maximum is 25 km per litre. Achieved at a speed of 45 km per hour.

Antiderivatives and integrals

We present the expected learning outcomes that we formulated followed by the sample diagnostic questions. The sample diagnostic questions are indicated under the following subheadings: anti-derivatives, indefinite integrals, and definite integrals.

Learning outcomes

We believe a student should be able to:

1. recall the defining conditions of an antiderivative of a function and be able to recall the symbol for antiderivative;
2. recall the antiderivative of standard functions;
3. recall laws of antiderivatives;
4. compute antiderivatives of new functions if the antiderivative is expressible in terms of standard functions;
5. recall the standard observations emerging from the antiderivative

- ➢ the derivative of the antiderivative gives the integrand
- ➢ any two antiderivatives of a function differs by a constant
- ➢ the area between the graph of a function on an interval and the x-axis is the difference of the values of the antiderivative of the function at the endpoints of the interval.

Diagnostic questions

This concerns the defining conditions for an antiderivative of a function, laws for integrals, integrals of standard and new functions, and the area interpretation of a definite integral.

Based on our literature review, conceptual framework and the learning outcomes arrived at we formulated the sample diagnostic questions indicated Tables 5, 6 and 7. For example our literature review on the suggestions of Abdul-Rahman (2005) implied that teaching needs to relate the antiderivative concept with that of the derivative, see Table 5 (Questions 1 & 2).

Table 5: Concept of the antiderivative

1.	What is the defining condition for a function $h(x)$ to be an antiderivative of a function $f(x)$?	$h'(x) = f(x)$
2.	Interpret $\int f(x)dx$	The general antiderivative (or integral) of the function $f(x)$ with respect to the variable x.

3.	Complete: Two antiderivatives of a function differ by a _______.	constant

Our formulations of the diagnostic questions in Table 6 were guided by the suggestions of Orton (1983) on the type of student errors with regard to the concept of integration. We also noted the work of Kiat (2005) who found students had difficulty with questions on (a) integration of trigonometric functions (see Question 5) and (b) applying integration to evaluate plane areas (see Table 7, Questions 7 & 10).

Table 6: Integration rules, applications

4.	For functions $f(x)$ and $g(x)$ complete the following: a) $\int kf(x)dx =$ b) $\int[f(x) \pm g(x)]\,dx =$	a) $k\int f(x)dx$ b) $\int f(x)\,dx \pm \int g(x)dx$
5.	If a and k are non-zero constant, complete the following integrals of standard functions: a) $\int kdx =$ b) For $n \neq -1$, $\int x^n dx =$ c) $\int x^{-1}dx -$ d) For $a > 0$ and $a \neq 1$, $\int e^{ax}dx =$ e) $\int a^{kx}\,dx =$ f) $\int \sin(ax)\,dx =$ g) $\int \cos(ax)\,dx =$ h) $\int \sec^2(ax)dx =$ i) $\int \csc^2(ax)\,dx =$ j) $\int \sec(ax)\tan(ax)\,dx =$ k) $\int \csc(ax)\cot(ax)dx =$	a) $kx + C$ b) $\frac{1}{n+1}x^{n+1} + C$ c) $\ln x + C$ d) $\frac{e^{ax}}{a} + C$ e) $\frac{a^{kx}}{k\ln a} + C$ f) $-\frac{1}{a}\cos(ax) + C$ g) $\frac{1}{a}\sin(ax) + C$ h) $\frac{1}{a}\tan(ax) + C$ i) $-\frac{1}{a}\cot(ax) + C$ j) $\frac{1}{a}\sec(ax) + C$ k) $-\frac{1}{a}\csc(ax) + C$
6.	Compute the following integrals: a) $\int\left(2 + x^3 - \frac{3}{x}\right)dx$ b) $\int\left(4e^{3t} + 7^{-2t} - \sqrt{t}\right)dt$	a) $2x + \frac{x^4}{4} - 3\ln x + C$ b) $\frac{4e^{3t}}{3} - \frac{7^{-2t}}{2\ln 7} - \frac{2}{3}t^{\frac{3}{2}} + C$ c) $ex - \frac{1}{e}\cos(ex) + C$

	c) $\int[e+\sin(ex)]dx =$ d) $\int \frac{\sin x}{1-\sin^2 x}dx =$	d) $\int \sec x \tan x\, dx = \sec x + C$

We arrived at the diagnostic questions for definite integrals indicated in Table 7 by focusing on the following from the literature review: A student's use of area under a curve is helpful in problem solving only when a *deeper understanding of the structure behind the definite integral is present* (Sealey, 2006), see Questions 7, 8 and 10. There should be a focus on the relationship between the graphical and symbolic integral representation, see Questions 7 and 10, since these could increase student performance of solving definite integral problems as agrued by Sevimli & Delice (2010).

Table 7: Definite integrals, rules and area interpretation

7.	Use the area interpretation of the definite integral to evaluate the following integrals: a) $\int_{-2}^{0}\sqrt{4-x^2}dx$ b) $\int_{-1}^{1}\lvert x\rvert dx$ c) $\int_{0}^{2\pi}\sin\theta d\theta$	a) π b) 1 c) 0
8.	State the conditions for $\int_a^b f(x)dx$ to exist.	The function $f(x)$ must be continuous on the closed $[a,b]$.
9.	State all the properties specific to definite integrals.	For continuous functions $f(x)$ and $g(x)$ on a closed interval $[a,b]$, and k a non-zero constant the following hold: g) $\int_a^a f(x)dx = 0$ h) $\int_a^b f(x)dx = -\int_b^a f(x)dx$ i) $\int_a^b kf(x)dx = k\int_a^b f(x)dx$ j) For any c in $[a,b]$, $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

		k) $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)$ l) For an even function continuous on $[-b, b]$, $\int_{-b}^{b} f(x)dx = 2\int_{0}^{b} f(x)dx$ m) For an odd function continuous on $[-b, b]$, $\int_{-b}^{b} f(x)dx = 0.$
10.	Evaluate the following definite integrals: a) $\int_0^{\frac{\pi}{2}} \cos x\, dx$ b) $\int_{-\pi}^{\pi} \sin x\, dx$	a) 1 b) 0

Creative thinking

We now present the learning outcomes and sample diagnostic questions that we arrived at. In our opinion this aspect is often neglected in the teaching and learning situation.

Learning outcomes

We expect students to be able to:

1. frame non-routine questions;
2. identify applications of mathematical concepts studied to various contextual situations in their surroundings.

Diagnostic questions

This concerns framing of new questions about the surrounding that seek to apply the content of the course.

1.	Find situations to apply the knowledge of the derivative in the context of cooking.	Indicative answer: Deep frying fish or chips. Rate of heat transfer from flame to pot, pot to oil, oil to interior of fish or chips. This is the reason for specific

		recipes indicating cooking time and the temperature at which cooking should occur. Note if the rate of heat transfer to oil is too high as compared the rate of heat from the oil to the interior of the fish then this results in the exterior of the fish getting burnt while the interior remains raw.
2.	Find situations to apply the knowledge of the derivative in the context of trading.	Indicative answer: Rate of supply, rate of distribution, rate demand and rate of financing govern the profitability of trade.
3.	Find situations to apply the knowledge of the derivative in the context of your other fields of study.	
4.	Find situations to apply the knowledge of the derivative in the context of classroom learning.	

CONCLUSIONS

The learning outcomes and the sample diagnostic questions that we arrived at for derivatives and integrals gave us insight into how the teaching and learning of these sections could be approached. In our opinion we documented material that could benefit both students and instructors. We will now plan for the implementation of these materials online, so that we could investigate how the material arrived at impacts on the teaching and learning of our students. Others who feel the material could be beneficial are encouraged to do the same. We will be interested in the experiences and finding of others.

ACKNOWLEDGEMENTS

Professor Sanjay Wagh from the Central India Research Institute in Nagpur (India) is acknowledged for facilitating my visits to Nagpur. Vivek Wagh from the Centre for Scientific Learning in Nagpur (India) for his advice to work on diagnostic testing for undergraduate mathematics and his help in formulating diagnostic questions. Mathematics' lecturers at UKZN for the use of past examination and test papers to inform the formulation of learning outcomes and diagnostic questions. This study was funded by the following grants: ESKOM's Tertiary Education Support Programme (TESP) for the *UKZN-ESKOM Mathematics Project*; the International Society for Technology Education for the HP Catalyst Multiversity Consortium project at UKZN *Mathematics e-Learning and Assessment: A South African Context*; the National Research Foundation in South Africa for the collaborative project headed by the author *Online diagnostics for undergraduate mathematics.*

REFERENCES

Abdul-Rahman S 2005. *Learning with examples and students' understanding of Integration.* Retrieved from http://math.unipa.it/~grim/21_project/21_malasya_AbdulRahman24-28_05.pdf

Adam S 2006. An introduction to Learning outcomes. Retrieved on 7/02/2013 from http://books.google.co.in/books?id=mxNFkhZXUisC&printsec=frontcover&source=gbs_ge_summary_r&cad=0#v=onepage&q&f=false

Betts JR, Hahn Y & Zau AC 2011. Does diagnostic math testing improve student learning? Retrieved on 7/02/2013 from http://mdtp.ucsd.edu/pdf/PPIC_MDTP_StudentLearning.pdf

Clark JM, Cordero F, Cottrill J, Czarnocha B, DeVries DJ, St. John D, Tolias T & Vidakovic D 1997. Constructing a schema: The case of the chain rule. *Journal of Mathematical Behavior*, 1:345-364.

Cottrill J 1999. *Students' understanding of the concept of chain rule in first year calculus and the relation to their understanding of composition of functions*. Phd thesis. Indiana: Purdue University.

Council of Regional Accrediting Commissions 2004. Regional accreditation and student learning: A guide for institutions and evaluators. Atlanta: Southern Association of Colleges & Schools. Available online: http://www.sacscoc.org/pdf/handbooks/GuideForInstitutions.PDF

Kiat S E 2005. Analysis of students' difficulties in solving integration problems. *The Mathematics Educator* , 9(1), 39-59. Accessed on 6/03/2013 at http://math.nie.edu.sg/ame/matheduc/tme/tmeV9_1/Seah%20EK.pdf

Learning and Technology Support Network Maths Team Project 2003. *Diagnostic testing for mathematics.* Retrieved on 7/02/2013 from *http://www.mathstore.ac.uk/headocs/diagnostic_test.pdf*

Giraldo V, Carvalho LM & Tall DO 2003. Descriptions and Definitions in the Teaching of Elementary Calculus. In NA Pateman, BJ Dougherty & J Zilliox (eds). Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education, 2:445-452. Honolulu, HI: Center for Research and Development Group, University of Hawaii. Available at http://www.warwick.ac.uk/staff/David.Tall/ pdfs/dot2003d-giraldo-carv-pme.pdf. Accessed 19 April 2012.

Gordon SP 2005. Discovering the chain rule graphically. *Mathematics and Computer Education*, 39:195-197.

Haciomeroglu ES, Aspinwall L & Presmeg N 2009. The role of reversibility in the learning of the calculus derivative and antiderivative graphs. In Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.), *Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Atlanta, GA: Georgia State University.

Hähkiöniemi M 2004. Perceptual and symbolic representations as a starting point of the acquisition of the derivative. *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, 3:73-80.

Hassani S 1998. Calculus students‟ knowledge of the composition of functions and the chain rule. Unpublished doctoral dissertation. Normal: Illinois State University.

Horvath A 2007. *Looking at calculus students' understanding from the inside-out: The relation-ship between the chain rule and function composition*. Available at http://sigmaa.maa.org/rume/crume2008/Proceedings/Horvath%20SHORT.pdf.Accessed 19 April 2012.

Maharaj A 2010. An APOS Analysis of Students' Understanding of the Concept of a Limit of a Function. *Pythagoras*, 71, July 2010, pp.41-52.

Maharaj A 2013. An APOS analysis of natural science students' understanding of derivatives**.** *South African Journal of Education*, 33(1) , 146-164.

Maharaj A 2014. An APOS Analysis of Natural Science Students' Understanding of Integration. *REDIMAT – Journal of Research in Mathematics Education*, *3*(1), 53-72.

Mahir N 2010. Students' Interpretation of a Function Associated with a Real-Life Problem from its Graph. PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies, 20(5), 392-404.

Orton A 1983. Students' Understanding of Differentiation. *Educational Studies in Mathematics*, 14:235-250.

Orton A 1983. Students' understanding of integration. *Educational Studies in Mathematics*, 14, 1-18.

Roorda G, Vos P & Goedhart M 2009. *Derivatives and applications; development of one student's understanding*. Proceedings of CERME 6, January 28th-February 1st 2009, Lyon France. Working group 12. Available at www.inrp.fr/editions/cerme6. Accessed 18 October 2010.

Sealey V 2006. Definite integrals, Riemann sums, and area under a curve: What is necessary and sufficient? In Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds), *Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol.2, p.46-53). Mérida, México:Universidad Pedagógica Nacional.

Sevimli E & Delice A 2010. The influence of teacher candidates' spatial visualization ability on the use of multiple representations in problem solving of definite integrals: A qualitative analysis http://www.bsrlm.org.uk/IPs/ip30-2/BSRLM-IP-30-2-10.pdf

Tall D 1993. *Students' Difficulties in Calculus*. Plenary Address. Proceedings of Working Group 3 on Students' Difficulties in Calculus, ICME-7, Québec, Canada, 13–28. Available at http://homepages.warwick.ac.uk/staff/David.Tall/downloads.html. Ac-cessed 18 October 2010.

Tall D 1997. Functions and Calculus. In AJ Bishop et al (eds). *International Handbook of Mathematics Education*. Dordrecht: Kluwer. Available at http://homepages.warwick.ac.uk/staff/David.Tall/pdfs/dot1997a-functions-calculus.pdf. Accessed 18 October 2010.

Tall D 2010. *A sensible approach to the calculus*. (Presented as a plenary at The National and International Meeting on the Teaching of Calculus, 23–25th September 2010, Puebla, Mexico.) Available at http://homepages.warwick.ac.uk/staff/David. Tall/downloads.html. Accessed 19 April 2012.

The California State University 2012. *Mathematics Diagnostic Testing Project*. Retrieved on 2/02/2013 from http://mdtp.ucsd.edu/OnLineTests.shtml

Ubuz B 1993. *Students' Understanding of Differentiation and Integration*. Retrived on 25 October 2011from http://www.bsrlm.org.uk/IPs/ip13-3/BSRLM-IP-13-3-12.pdf

Uygur T & Özdaş A 2005. Misconceptions and difficulties with the chain rule. In *The Mathematics Education into the 21st century Project*. Malaysia: University of Teknologi. Available at http://math.unipa.it/~grim/21_project/21_malasya_Uygur209-213_05.pdf. Accessed 18 October 2010.

Uygur T& Özdaş A 2007. The effect of arrow diagrams on achievement in applying the chain rule. *Primus: Problems, Resources, and Issues in Mathematics Undergraduate Studies*, 17:131-147.

Webster RJ 1978. The effects of emphasizing composition and decomposition of various types of composite functions on the attainment of chain rule application skills in calculus. Unpublished doctoral dissertation. Tallahassee: Florida State University.

Zandieh MJ 2000. A Theoretical Framework for Analyzing Student Understanding of the Concept of Derivative. *CBMS Issues in Mathematics Education*, 8:103-122.